The converse of Léon Anne's theorem

ANDREW JOBBINGS
www.arbelos.co.uk

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Theorem 1 (Léon Anne's theorem) *Pick a point P in the interior of a quadrilateral (which is not a parallelogram). Join this point to each of the four vertices. Then the locus of points P for which the sum of opposite triangle areas is half the quadrilateral area is the line joining the midpoints of the diagonals (see figure 1).* □

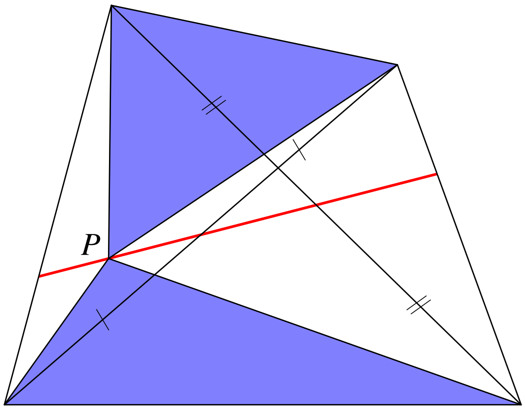

Figure 1

We provide a proof of the *converse* result. In other words, we show that the points on the line joining the midpoints of the diagonals have the required area property. Of course, to prove the theorem it is also necessary to show that these are the *only* such points.

Our proof makes use of two lemmas.

Lemma 1 *In triangle PXY, let M be the midpoint of XY. Then* area PXM = area PMY. □

PROOF See figure 2. The two triangles PXM and PMY have equal bases, since $XM = MY$, and the same height. Therefore their areas are equal. ■

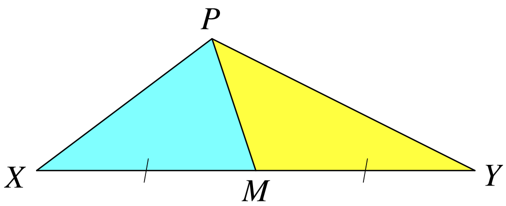

Figure 2

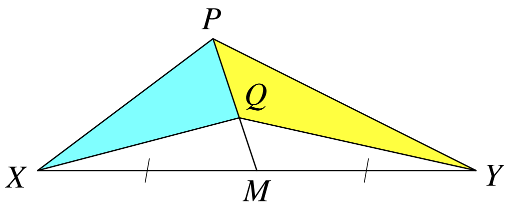

Figure 3

Lemma 2 *In triangle PXY, let M be the midpoint of XY and let Q be a point on PM. Then* area PXQ = area PQY. □

PROOF See figure 3. From lemma 1, we know that

$$\text{area}\, PXM = \text{area}\, PMY$$
$$\text{and} \quad \text{area}\, QXM = \text{area}\, QMY.$$

Hence

$$\text{area}\, PXQ = \text{area}\, PXM - \text{area}\, QXM = \text{area}\, PMY - \text{area}\, QMY = \text{area}\, PQY.$$ ■

Theorem 2 (Converse of Léon Anne's theorem) *In the quadrilateral $ABCD$ (which is not a parallelogram), let E and F be the midpoints of the diagonals AC and BD, and let P be any point inside the quadrilateral on the line EF (see figure 4).*

Then

$$\text{area } PBC + \text{area } PDA = \tfrac{1}{2} \text{ area } ABCD.$$

□

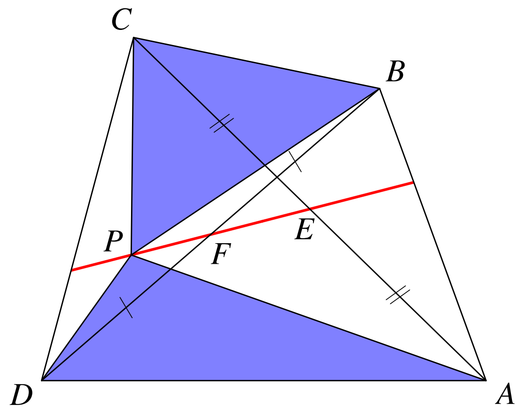

Figure 4

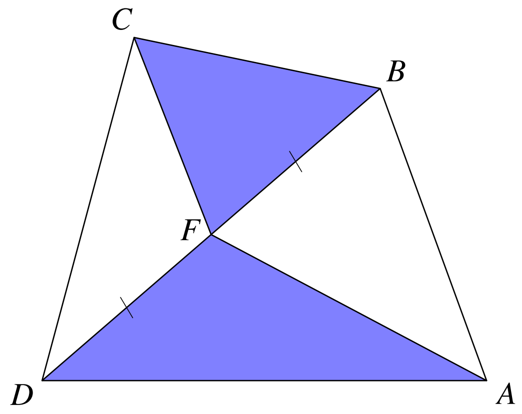

Figure 5

Proof We first show that F is one possible position of P (see figure 5). From lemma 1 we have

$$\text{area } FAB = \text{area } FDA$$
$$\text{and} \quad \text{area } FCD = \text{area } FBC,$$

so that

$$\text{area } FAB + \text{area } FCD = \text{area } FBC + \text{area } FDA.$$

It follows that

$$\text{area } FBC + \text{area } FDA = \tfrac{1}{2} \text{ area } ABCD. \tag{1}$$

Now we turn our attention to P.

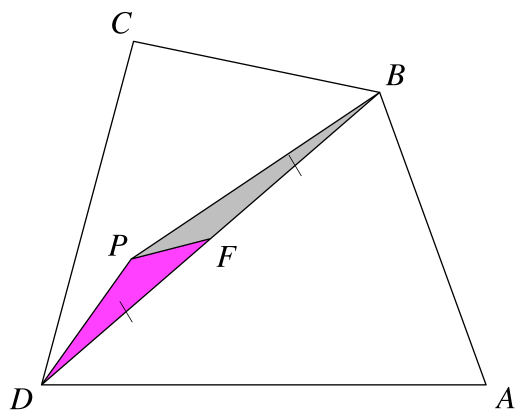

Figure 6

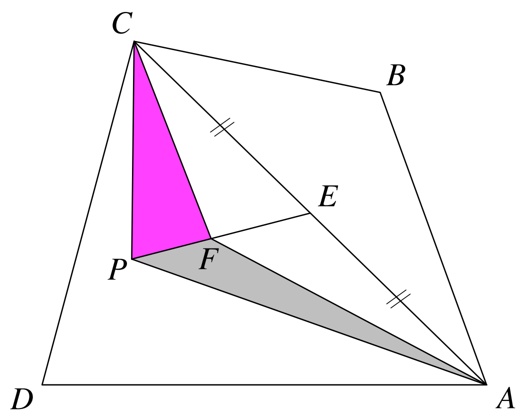

Figure 7

Using lemmas 1 and 2 in figures 6 and 7, we obtain

$$\text{area } PFD = \text{area } PFB$$
$$\text{and} \quad \text{area } PFC = \text{area } PFA,$$

so that

$$\text{area}\, PFC + \text{area}\, PFD = \text{area}\, PFA + \text{area}\, PFB.$$

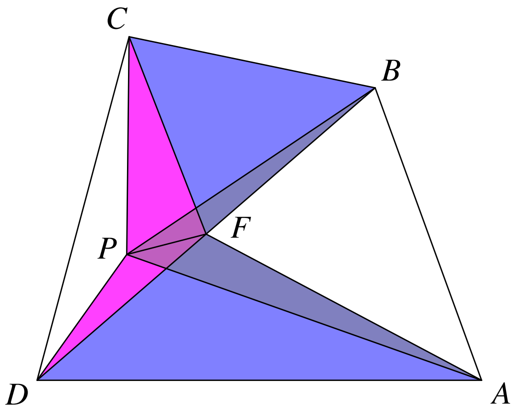

Figure 8

Therefore we have (see figure 8)

$$\begin{aligned}\text{area}\, FBC + \text{area}\, FDA &= \text{area}\, FBC + \text{area}\, FDA \\ &\quad + (\text{area}\, PFC + \text{area}\, PFD) \\ &\quad - (\text{area}\, PFA + \text{area}\, PFB) \\ &= \text{area}\, PBC + \text{area}\, PDA\end{aligned}$$

and the required result follows from equation (1). ■

Notes

1. The above proof is 'diagram dependent'—it assumes, for example, that P lies on EF produced. However, it is straightforward to modify the proof to deal with other cases.
2. What happens if the quadrilateral is not convex?
3. If the quadrilateral is a parallelogram, then the midpoints of the diagonals are the same point, so the line EF is undefined. In this case it is possible to prove that *any* point P inside the quadrilateral has the required area property.
4. The line EF is sometimes called the *Newton-Gauss line* of the quadrilateral. It may be shown that the Newton-Gauss line also passes through the midpoint of the line segment joining the two points where opposite sides of the quadrilateral meet.
5. A proof by vector products is given in [1].

References

[1] Chris Pritchard. 'Generalizing Complementary Regions'. In: *Mathematics in School* 42.2 (Mar. 2013), pp. 11–13.