British Mathematical Olympiad 2004/5

Round 1: question 5

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Question 5

Let \( S \) be a set of rational numbers with the following properties:

i) \( \frac{1}{2} \in S \);

ii) if \( x \in S \), then both \( \frac{1}{x+1} \in S \) and \( \frac{x}{x+1} \in S \).

Prove that \( S \) contains all rational numbers in the interval \( 0 < x < 1 \).

Continued fractions

Use the notation \([a_1, a_2, \ldots, a_n]\) for the continued fraction

\[
\frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \cdots + \frac{1}{a_n}}}}
\]

where the \( a_i \) are positive integers. Note that \([a_1, a_2, \ldots, a_{n-1}, 1] = [a_1, a_2, \ldots, a_{n-1} + 1]\), so we may assume that \( a_n \geq 2 \).

To find the continued fraction of a given fraction, repeatedly invert and divide. For example, to find the continued fraction of \( \frac{28}{39} \) proceed as follows:

\[
\frac{28}{39} \rightarrow \frac{39}{28} = 1 + \frac{11}{28} \rightarrow \frac{28}{11} = 2 + \frac{6}{11} \rightarrow \frac{11}{6} = 1 + \frac{5}{6} \rightarrow \frac{6}{5} = 1 + \frac{1}{5} \rightarrow \frac{5}{1} = 5.
\]

Hence we have \( \frac{28}{39} = [1, 2, 1, 1, 5] \).

Solution to question 5

Let \( f(x) = \frac{1}{x+1} \) and \( g(x) = \frac{x}{x+1} \). Then \( gf(0) = \frac{1}{2} \) and

\[
g^{r-1}f(x) = \frac{1}{r + x}.
\]

Hence

\[
[a_1, a_2, \ldots, a_n] = g^{a_1-1}f \circ g^{a_2-1}f \circ \cdots \circ g^{a_n-1}f(0)
\]

\[
= g^{a_1-1}f \circ g^{a_2-1}f \circ \cdots \circ g^{a_n-2}(\frac{1}{2})
\]
which gives a closed form for \([a_1, a_2, \ldots, a_n]\) in terms of \(f\) and \(g\).

It follows that any rational number in the interval \(0 < x < 1\) belongs to \(S\), since any such rational may be expressed as a continued fraction of the form \([a_1, a_2, \ldots, a_n]\).

**Example**

For example,

\[
\frac{28}{39} = [1, 2, 1, 5] = f \circ g \circ f \circ f \circ g^3 \left( \frac{1}{2} \right)
\]

giving the sequence of fractions

\[
\frac{1}{2} \mapsto \frac{1}{3} \mapsto \frac{1}{4} \mapsto \frac{1}{5} \mapsto \frac{5}{6} \mapsto \frac{6}{11} \mapsto \frac{11}{17} \mapsto \frac{28}{39}.
\]