British Mathematical Olympiad 2004/5

Round 1: question 5

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Question 5

Let S be a set of rational numbers with the following properties:

i) $\frac{1}{2} \in S$;

ii) if $x \in S$, then both $\frac{1}{x+1} \in S$ and $\frac{x}{x+1} \in S$.

Prove that S contains all rational numbers in the interval $0 < x < 1$.

Continued fractions

Use the notation $[a_1, a_2, \ldots, a_n]$ for the continued fraction

$$\cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_n}}}}$$

where the a_i are positive integers. Note that $[a_1, a_2, \ldots, a_{n-1}, 1] = [a_1, a_2, \ldots, a_{n-1} + 1]$, so we may assume that $a_n \geq 2$.

To find the continued fraction of a given fraction, repeatedly invert and divide. For example, to find the continued fraction of $\frac{28}{39}$ proceed as follows:

$$\frac{28}{39} \to \frac{39}{28} = \mathbf{1} + \frac{11}{28} \to \frac{28}{11} = \mathbf{2} + \frac{6}{11} \to \frac{11}{6} = \mathbf{1} + \frac{5}{6} \to \frac{6}{5} = \mathbf{1} + \frac{1}{5} \to \frac{5}{1} = \mathbf{5}.$$

Hence we have $\frac{28}{39} = [1, 2, 1, 1, 5]$.

Solution to question 5

Let $f(x) = \frac{1}{x+1}$ and $g(x) = \frac{x}{x+1}$. Then $gf(0) = \frac{1}{2}$ and

$$g^{r-1} f(x) = \frac{1}{r + x}.$$

Hence

$$\begin{aligned}[a_1, a_2, \ldots, a_n] &= g^{a_1 - 1} f \circ g^{a_2 - 1} f \circ \cdots \circ g^{a_n - 1} f(0) \\ &= g^{a_1 - 1} f \circ g^{a_2 - 1} f \circ \cdots \circ g^{a_n - 2} \left(\tfrac{1}{2}\right)\end{aligned}$$

which gives a closed form for $[a_1, a_2, \ldots, a_n]$ in terms of f and g.

It follows that any rational number in the interval $0 < x < 1$ belongs to S, since any such rational may be expressed as a continued fraction of the form $[a_1, a_2, \ldots, a_n]$.

Example

For example,

$$\begin{aligned}\frac{28}{39} &= [1, 2, 1, 1, 5] \\ &= f \circ gf \circ f \circ f \circ g^3\left(\tfrac{1}{2}\right)\end{aligned}$$

giving the sequence of fractions

$$\frac{1}{2} \overset{g}{\longmapsto} \frac{1}{3} \overset{g}{\longmapsto} \frac{1}{4} \overset{g}{\longmapsto} \frac{1}{5} \overset{f}{\longmapsto} \frac{5}{6} \overset{f}{\longmapsto} \frac{6}{11} \overset{f}{\longmapsto} \frac{11}{17} \overset{g}{\longmapsto} \frac{11}{28} \overset{f}{\longmapsto} \frac{28}{39}.$$