

Chords and regions

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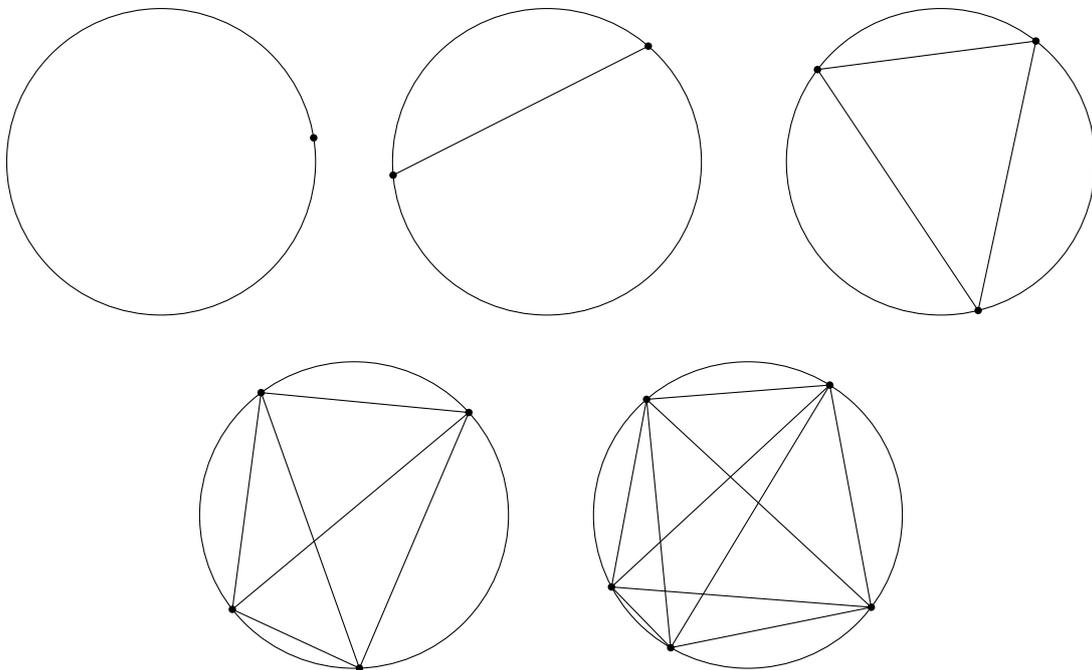
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Place n points on a circle and draw all the chords which connect the points. Suppose that no three chords meet inside the circle (that is, the chords are in *general position*). How many regions are formed inside the circle?

Pattern spotting

For the first few values of n we get the following figures.



Counting the regions, we obtain the following table.

Points	Regions
1	1
2	2
3	4
4	8
5	16

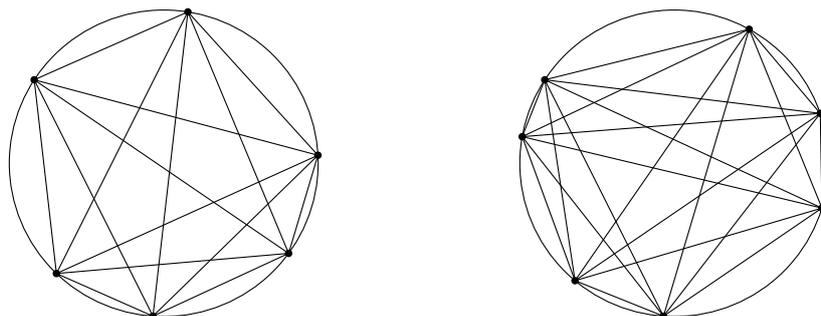
Notice that the numbers of regions in the table are all powers of 2:

$$1 = 2^0; \quad 2 = 2^1; \quad 4 = 2^2; \quad 8 = 2^3 \quad \text{and} \quad 16 = 2^4.$$

Having ‘spotted’ this pattern, the temptation now is to declare that the number of regions is always a power of 2, and to state that for n points there are 2^{n-1} regions. Unfortunately, there are two difficulties with this approach:

1. The statements have not been justified. How do we know that the pattern will continue for every value of n ?
2. The statements are *wrong*!

Though the second of these points applies in this example—showing that spotting a pattern is risky even when just trying to discover the form of a result—the first point is the more important one. Whatever the context, ‘pattern spotting’ alone can never prove anything, and such an approach is never appropriate in a mathematical proof without some rigorous justification that the pattern continues for ever.



To see that the proposed statements are incorrect, consider the case $n = 6$, shown on the left above. There are 31 regions in the figure, whereas $2^{6-1} = 2^5 = 32$. The figure on the right above verifies that the problem continues when $n = 7$, where there are 57 regions.

The actual formula

To find the actual formula for the number of regions we have to work harder.

Result *The number of regions formed is*

$$1 + \binom{n}{2} + \binom{n}{4}. \quad \square$$

The two terms with brackets are binomial coefficients, which means that

$$\binom{n}{2} = \frac{n(n-1)}{2 \times 1} \quad \text{and} \quad \binom{n}{4} = \frac{n(n-1)(n-2)(n-3)}{4 \times 3 \times 2 \times 1}.$$

Using these expressions it is possible to find a polynomial formula for the number of regions, namely $\frac{1}{24}(n^4 - 6n^3 + 23n^2 - 18n + 24)$.

Instead, we shall give two proofs of the result by using the fact that the binomial coefficient $\binom{n}{r}$ is equal to the number of ways of choosing r objects from a set of n (different) objects.

PROOF 1 We shall count the number of regions by considering drawing the chords one by one.

When it is drawn each new chord crosses a number of regions, dividing each of them into two. The number of extra regions created is thus equal to the number of regions crossed, which is one more than the number of chords crossed. Since the new chord cannot pass through a previously drawn point of intersection, the number of chords crossed is equal to the number of

interior points of intersection on the new chord. Hence the number of extra regions created is one plus the number of interior points of intersection on the new chord

Therefore the total number of extra regions obtained by drawing all the chords is equal to the number of chords (the total of the ‘ones’) added to the total number of interior points of intersection.

How many chords are there? Each chord is determined by two points on the circle, and there are $\binom{n}{2}$ ways to choose two of these n points. Hence there are $\binom{n}{2}$ chords.

How many points of intersection are there inside the circle? Each is determined by choosing four of the n points on the circle, and there are $\binom{n}{4}$ ways to do this. Hence there are $\binom{n}{4}$ interior points of intersection.

So the number of extra regions obtained by drawing all the chords is $\binom{n}{2} + \binom{n}{4}$. But there is one region to start with (the interior of the circle), so that

$$\text{the number of regions} = 1 + \binom{n}{2} + \binom{n}{4}. \quad \blacksquare$$

A more sophisticated proof uses Euler’s formula to find a relation between the number of points, lines and regions in the diagram. To do this we consider the diagram to be a graph in the plane, whose vertices are the original points together with the points of intersection of the chords.

Euler’s formula *Let V be the number of vertices, E the number of edges and F the number of faces of a planar graph. Then $V - E + F = 2$.*

PROOF 2 (USING EULER’S FORMULA) The number of ‘faces’ is equal to the number of regions, except that there is also a face formed by the region outside the circle.

To find the number of ‘vertices’, we first note that there are n on the circle—the original points. As we found in the first proof above there are $\binom{n}{4}$ points of intersection inside the circle, so there are $n + \binom{n}{4}$ vertices altogether.

To find the number of ‘edges’, first note that there are n edges which are circular arcs. We count the other edges by dealing separately with interior points and points on the circle.

Four edges meet at each of the $\binom{n}{4}$ interior vertices, making $4\binom{n}{4}$ edges. There are $\binom{n}{2}$ chords, as shown in the first proof above, and each chord corresponds to two edges meeting the circle, making $2\binom{n}{2}$ edges. We have thus counted $2\binom{n}{2} + 4\binom{n}{4}$ edges, but each of them has been counted twice in this process, once for each of the vertices at its ends.

Hence there are $n + \binom{n}{2} + 2\binom{n}{4}$ edges altogether.

We now have expressions for the numbers of faces, vertices and edges. Substituting these expressions into Euler’s formula, we get

$$\left\{ n + \binom{n}{4} \right\} - \left\{ n + \binom{n}{2} + 2\binom{n}{4} \right\} + \{ \text{the number of regions} + 1 \} = 2,$$

which rearranges to give

$$\text{the number of regions} = 1 + \binom{n}{2} + \binom{n}{4}. \quad \blacksquare$$