Dissections

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Simple dissections

There are some familiar examples where a method of dissection is used to prove that the areas of two different shapes are equal. For example, if the parallelogram in Figure 1 is cut into two equal parts these may be reassembled to form a rectangle, as shown. This is one way of deriving the expression *base* × *height* for the area of a parallelogram.

Figure 1: Dissection of a parallelogram to a rectangle

The idea of dissecting and rearranging can be useful when attempting challenge problems. A question on the Kangaroo 2000 paper asked for the shaded area in the diagram on the left of Figure 2. One way to do this is to dissect the shape [Figure 2, centre] and rearrange [Figure 2, right], so that the area may be easily determined. The correct answer is therefore . . . ?

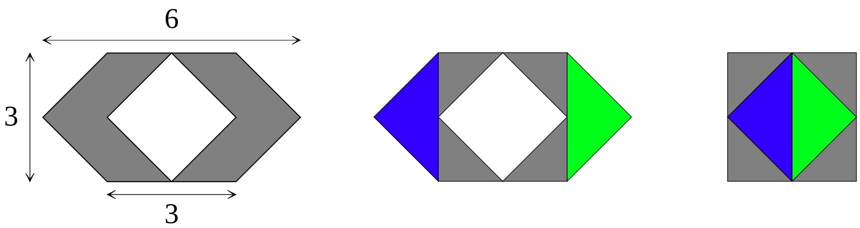

Figure 2: Using dissection to solve a Kangaroo question

Another example of a slightly different type occurred on the JMO 1999 paper, which asked what fraction of the whole square was occupied by the shaded square in the diagram on the left of Figure 3. Dissecting [Figure 3, centre] and rearranging [Figure 3, right] the large square as indicated shows that the grey square occupies one tenth of the whole area.

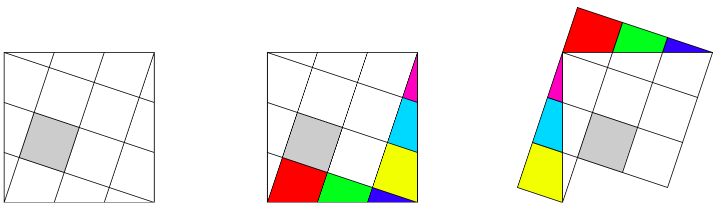

Figure 3: Using dissection to solve a JMO question

Dissections and superimposed tessellations

Where did the dissection in Figure 3 come from? A clue is provided if five copies of the original diagram are placed together as shown in Figure 4. This diagram not only verifies that the given dissection and rearrangement are exact, but also shows that the original diagram may be seen as part of a larger arrangement, obtained by superimposing two different grids of squares.

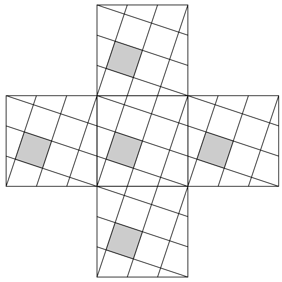

Figure 4: Tessellating the dissection of Figure 3

Now, grids of squares are examples of tessellations of the plane. Extending this idea to other tessellations leads to some further dissection results.

The American puzzlist Sam Loyd (1841–1911) devised the elegant problem: show how to dissect two Greek crosses to form a square. The solution is shown in Figure 5, on the left, and the corresponding superimposed tessellations on the right.

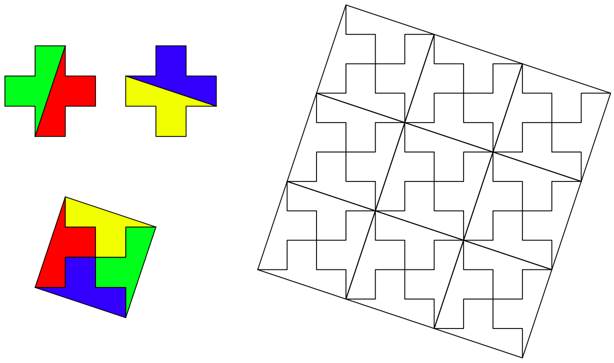

Figure 5: The solution to Sam Loyd's problem and the corresponding tessellations

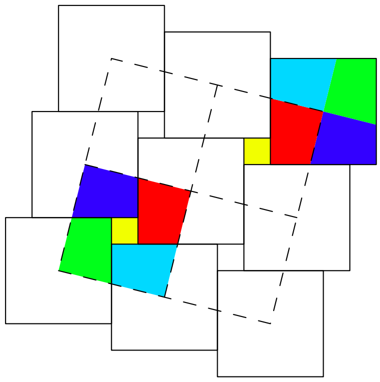

Figure 6: Perigal's dissection

Several dissection proofs of Pythagoras Theorem may be obtained in this way, including a well-known one by Henry Perigal (1873), shown in Figure 6.

General dissections

One might ask: given any two polygons of equal area, is it always possible to dissect one of them and rearrange the pieces to make the other? That this is the case was proved by William Wallace in 1807, though it is usually called the Bolyai-Gerwien Theorem (Wolfgang Bolyai asked the question and P Gerwien answered it in 1833).

Can you see how to dissect any given trapezium by a single straight cut into two pieces which can be rearranged to form a triangle?

References

Greg N. Frederickson. *Dissections Plain & Fancy*. Cambridge University Press, 2003.
David Wells. *The Penguin Book of Curious and Interesting Geometry*. Penguin, 1991.