Folding paper

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Introduction

When paper-folding is mentioned, your thoughts may immediately turn to origami and folding objects like birds or flowers. This artistic side of paper-folding is well-known, but it is less well known that there are mathematical aspects too, and many of them. The purpose of this article is to draw attention to some of this mathematics, and to show how folding paper may be helpful in teaching. Paper is ubiquitous, and it can be a useful and satisfying resource in the mathematics classroom. Our aim is to whet your appetite and point out some of the resources available.

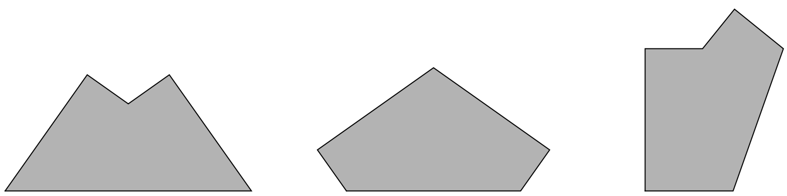
Note that traditional origami starts with a square piece of paper, but for obvious reasons we wish to work more generally with rectangular sheets, such as A4, so unless specified otherwise the starting sheet of paper will be assumed to be a general rectangle.

Sample problems

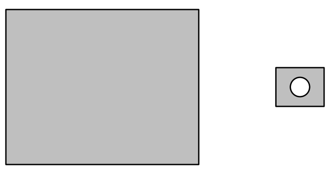
The mathematics involved in folding paper ranges from elementary upwards, and new research is still being undertaken. Topics include 'spatial awareness' and geometry, not surprisingly, but also combinatorics and logic. For more information see [13].

To appreciate the variety of mathematics that arises, you may like to try the following problems, selected from those I have devised for the [12].

Problem 1 (JMC 1999)
A sheet of A4 paper is folded once *and placed flat on a table. Which of the following shapes could not be made?*

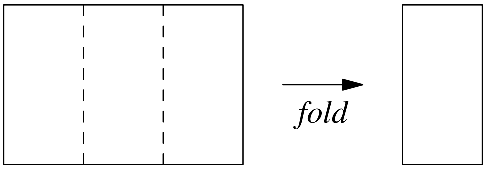


Problem 2 (JMC 2005)
Beatrix takes a sheet of paper (shown on the left), folds the sheet in half 4 times and punches a hole all the way through the folded sheet, as shown on the right. She then unfolds the sheet.



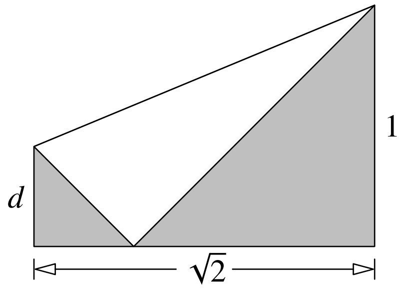
How many holes are there now in the unfolded sheet?

Problem 3 (JMC 2001)
The sheet of paper shown on the left is folded along the dotted lines (each fold being either forwards or backwards) to make the leaflet shown on the right. Each of the six 'pages' of the leaflet is printed in a different colour. No matter how it is folded, the leaflet will have two pages visible on the outside.



How many different pairs of outside pages can be obtained by folding the sheet of paper in different ways?

Problem 4 (IMC 1999)
A rectangular sheet of paper with sides 1 *and* $\sqrt{2}$ *has been folded once as shown, so that one corner meets the opposite long edge. What is the value of the length* d*?*



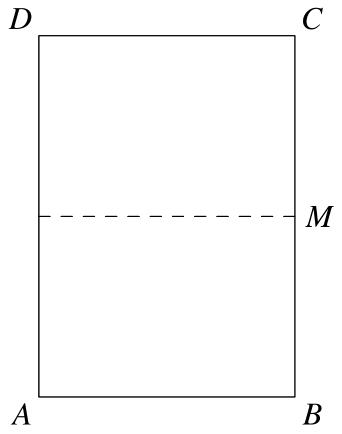
Simple shapes

On the way to learning about Euclidean geometry in a more formal way, pupils pass through a 'scientific' or experimental stage, where they use processes such as observation and classification to develop their understanding of geometrical objects. During this formative phase, practical methods like paper folding are very appropriate.

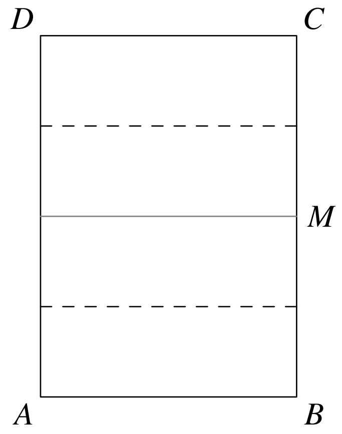
As an example, see the instructions for folding a rhombus given on the following page. This is one of several shapes included in the paper *How to fold simple shapes from A4 paper* [8], which is freely available and discusses the folding methods in more detail. Included there is a proof that the rhombus has angles of 60° and 120°. These special angles mean that folding the rhombus in half gives an equilateral triangle; alternatively, making two different folds creates a regular hexagon.

The 'model' may be used in the classroom to demonstrate some of the properties of a rhombus in a practical way—folding in half, say, to show that pairs of sides are equal, or that a diagonal bisects the angle at a vertex. This can all be done without measurement, since the lengths or angles actually coincide on the model. Similarly, showing that the diagonals are perpendicular just requires the two diagonals to be folded: can you see how to confirm, without measuring, that the angle between the diagonals is 90°?

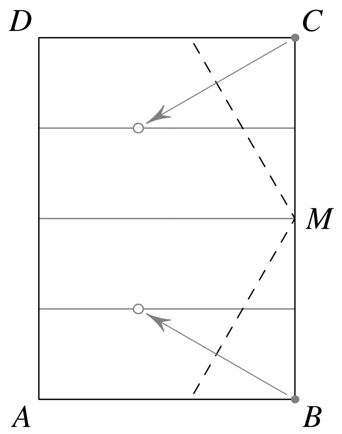
To fold a rhombus



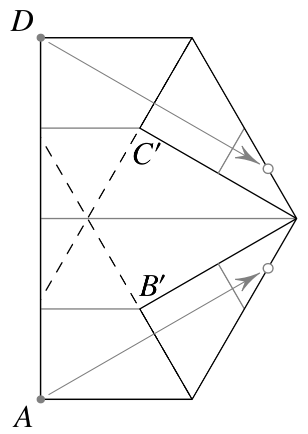
STEP 1
Fold a sheet of A4 paper in half widthwise and unfold.



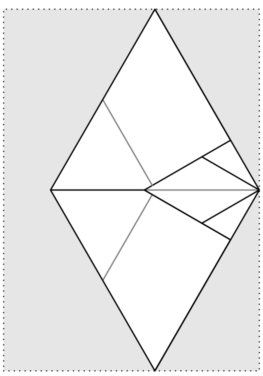
STEP 2
Fold AB and CD onto the crease from Step 1 and unfold.



STEP 3
Fold B onto the lower crease from Step 2 so that the new crease passes through M; repeat with C and the upper crease.



STEP 4
Fold A onto the lower crease from Step 3 so that the new crease passes through B'; repeat with D, the upper crease and C'.



The completed rhombus.

Folding in strips

"How many times can you fold a sheet of paper in half?" is a popular question. The answer, of course, depends a lot on the sheet of paper (and a little on you), but in 2002 a world record was set, by folding a sheet of paper 12 times! See [3].

The idea behind Problem 3 can be extended in various ways. Suppose you have a strip of, say, 6 stamps. In how many different ways can you fold the strip along the perforations (of course, you first need to decide what is meant by 'different')? Suppose that the stamps are labelled A, B, C, . . . , then which orderings of the stamps are achievable by folding? Now extend these questions from a strip to a rectangular sheet of stamps, to give the so-called map-folding problems. *Folding Stamps and Maps* [6] is a classroom activity based on these ideas. You should also have a look at [2]. (The Demaines are at the forefront of research in this field; see [1] for a quirky example.)

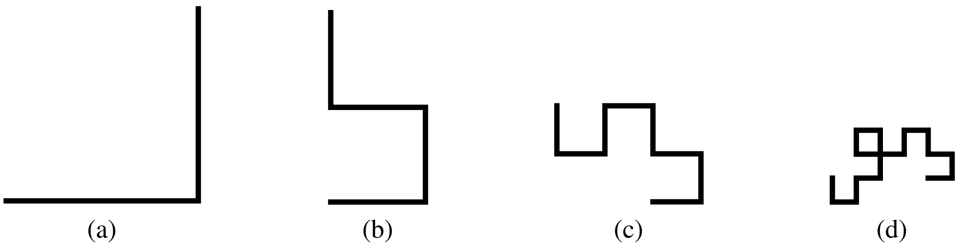


Figure 1: Unfolding a folded strip.

Take a long strip of paper, fold it in half (widthwise) and unfold so that the crease forms an angle of 90° (see figure 1a). If instead you fold the strip in half twice, then unfold in the same way, you obtain figure 1b. Continuing in this way, you obtain a sequence of 'curves': the first four iterations are shown in figure 1; the 12th in figure 2. The *dragon curve* is defined to be the limit of this sequence and has many interesting properties; it is, for example, a space-filling curve. The curve features in full colour on one of the posters in the set *Fractals* [7].

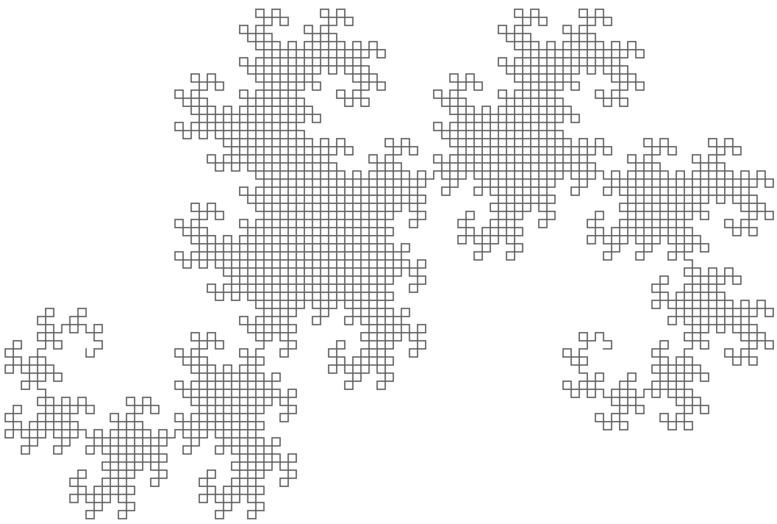


Figure 2: The dragon curve: 12th iteration.

Geometry

When you tried Problem 4 you probably made use of Pythagoras' theorem and it is perhaps not surprising that Pythagoras is involved for a configuration of this type. For another example, consider figure 3, which shows a *square* sheet folded so that the lower left corner meets the midpoint M of the top edge. Where are the fold points on the side edges? The required points may be obtained using Pythagoras' theorem. Observe that 3-4-5 triangles make an appearance!

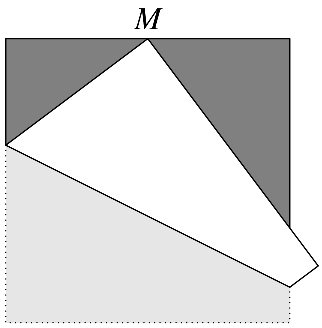


Figure 3: Folding a square.

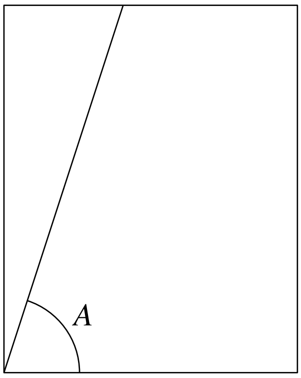
The result generalises: instead of dividing the top edge in the ratio $1 : 1$, consider the case where M divides the edge in the ratio $p : q$. The resulting configuration is known as Haga's construction, which may be used, repeatedly, to divide the side of a square in any ratio. See Lang's article 'Origami and Geometric Constructions' [9] for a full discussion of this and similar results.

Anyone familiar with classical 'straightedge and compass' constructions may be surprised to learn that everything that can be constructed in the classical way may also be achieved by folding paper, indeed, even more is possible by paper-folding than by classical constructions. For example, a given angle may be trisected (see figure 4), something that is not possible with straightedge and compass. Angle trisection is incorporated in the classroom activity *Folding Curves and Thirds* [5]; a proof of the construction is given in the accompanying Teacher's Notes.

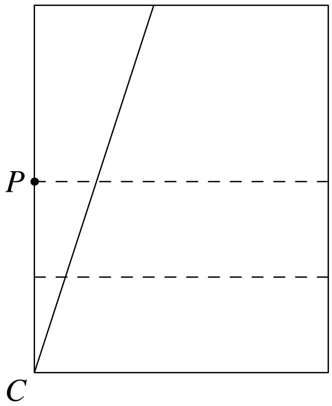
The book *Geometric Origami* [4] includes a full discussion of paper-folding constructions and gives folding instructions for several regular polygons, such as the heptagon, known not to be constructible in the traditional way.

Applications

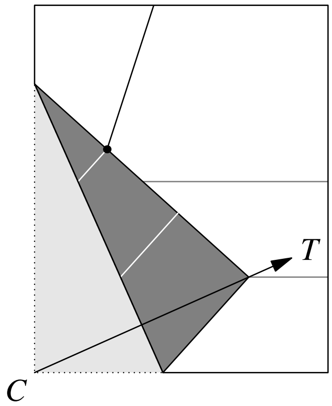
The beginning of this article referred to origami as the artistic side of paper-folding. Recently the use of mathematics—specifically, ideas related to circle packing—has led to significant developments in origami art, by helping to answer the question "how do you create a crease pattern for a given origami object?" The same work has led to developments in engineering and medicine too, amongst other fields. For a description of applications to stents and space telescopes, for example, see [11]. A remarkable 16-minute video [10] describes several applications: to a TV advert, telescopes, airbags and stents, as well as to origami itself.



(a) The given angle A.



(b) Fold up bottom edge to make two equally spaced creases.



(c) Fold up so that C lies on first crease and P lies on given line. CT is the line of trisection.

Figure 4: Trisecting an angle.

References

[1] Devin J. Balkcom et al. 'Folding Paper Shopping Bags'. In: *Origami⁴: Proceedings of the 4th International Meeting of Origami Science, Math, and Education*. A K Peters, 2006, pp. 315–334. URL: http://erikdemaine.org//papers/PaperBag_OSME2006/.

[2] Erik Demaine and Martin Demaine. *Erik Demaine and Martin Demaine's Puzzles*. 2010. URL: http://erikdemaine.org/puzzles/.

[3] *Folding Paper in Half 12 Times*. The Historical Society of Pomona Valley. URL: http://www.pomonahistorical.org/12times.htm.

[4] Robert Geretschläger. *Geometric Origami*. Arbelos, 2008. ISBN: 978-0-9555477-1-3. URL: http://www.arbelos.co.uk/GeometricOrigami.html.

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[9] Robert J. Lang. 'Origami and Geometric Constructions'. 2003. URL: http://www.langorigami.com/science/hha/hha.php4.

[10] Robert J. Lang. *Robert Lang folds way-new origami*. TED. 2008. URL: http://www.ted.com/index.php/talks/robert_lang_folds_way_new_origami.html.

[11] Tom Shelley. *Expanding structures reach out to save lives*. 2005. URL: http://www.eurekamagazine.co.uk/article/3274/Expanding-structures-reach-out-to-save-lives.aspx.

[12] *United Kingdom Mathematics Trust*. URL: http://www.mathcomp.leeds.ac.uk/.

[13] Wikipedia contributors. *Mathematics of paper folding*. URL: http://en.wikipedia.org/wiki/Mathematics_of_paper_folding.

Answers

Problem 1: all are possible.
Problem 2: 16.
Problem 3: 6.
Problem 4: $\sqrt{2} - 1$.