Geometry by numbers

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Introduction

There are some geometrical results which are difficult to prove by purely Euclidean methods and approaches involving coordinate geometry or trigonometry are not always appropriate or may lead to messy algebraic manipulation. An alternative approach is to use complex numbers and for some results this may be the most convenient method of proof. In this article we demonstrate the method by looking at a few results about polygons.

That squares feature here is perhaps not surprising, since multiplication by i corresponds to a rotation through $90°$. However, as we shall see, complex numbers may also help with results involving regular polygons.

Notation

Complex numbers are represented as points in an Argand diagram—the complex plane. We use uppercase letters for the labels of geometric points and lowercase letters for the corresponding complex numbers, so p is the complex number corresponding to point P. In particular, the point O corresponds to the number 0.

Basic results

Complex numbers may be considered to correspond to vectors: the number z corresponds to the vector $\overrightarrow{OZ}$ joining 0 to z. Also, if $w = iz$, then $\overrightarrow{OW}$ is obtained from $\overrightarrow{OZ}$ by rotating through $90°$ about O (by convention positive angles are anticlockwise). More generally, we have:

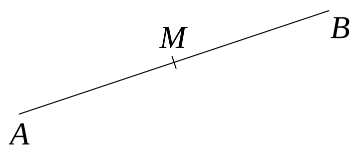
Property 1 *The number $w - z$ corresponds to the vector $\overrightarrow{ZW}$.*

Property 2 *Multiplying a number by i rotates the corresponding vector through* $90°$.

These properties are key to the methods used here. We state them without further discussion: most standard texts will provide derivations and more detail.

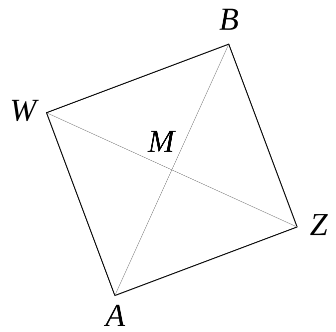
Examples

1. Suppose M is the midpoint of AB.



Then $\overrightarrow{AB} = 2\overrightarrow{AM}$ and so $b - a = 2(m - a)$. Therefore $m = \frac{1}{2}(a + b)$.

2. The distance from Z to W, that is, the length ZW, is equal to the modulus $|w - z|$.
3. If $b - a = i(q - p)$, then AB and PQ are perpendicular and equal in length.
4. Suppose $ABCD$ is a square (labelled anticlockwise). Then $d - a = i(b - a)$.
5. Given two complex numbers a, b, what are the values of z, w so that $AZBW$ is a square?



Let M be the midpoint of AB. Since MZ is perpendicular to BA and of half the length we have

$$z - m = \tfrac{1}{2}i(a - b)$$

and hence

$$z = \tfrac{1}{2}(a + b) + \tfrac{1}{2}i(a - b).$$

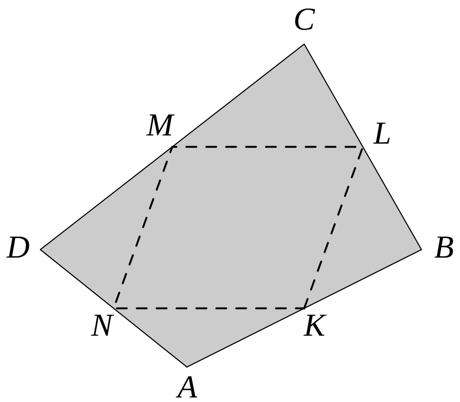
Similarly,

$$w = \tfrac{1}{2}(a + b) - \tfrac{1}{2}i(a - b).$$

Quadrilaterals

Though the following theorem is not difficult to prove by Euclidean methods, for example by using the midpoint theorem, we give a proof using complex numbers in order to show how to apply the above ideas.

Varignon's theorem *The midpoints of the sides of an arbitrary planar quadrilateral form a parallelogram.*



Let the midpoints be K, L, M and N, as shown. Then using the result of Example 1 we have $k = \frac{1}{2}(a + b)$, with similar results for l, m and n. Therefore

$$\begin{aligned} l - k &= \tfrac{1}{2}(b + c) - \tfrac{1}{2}(a + b) \\ &= \tfrac{1}{2}(c - a) \end{aligned}$$

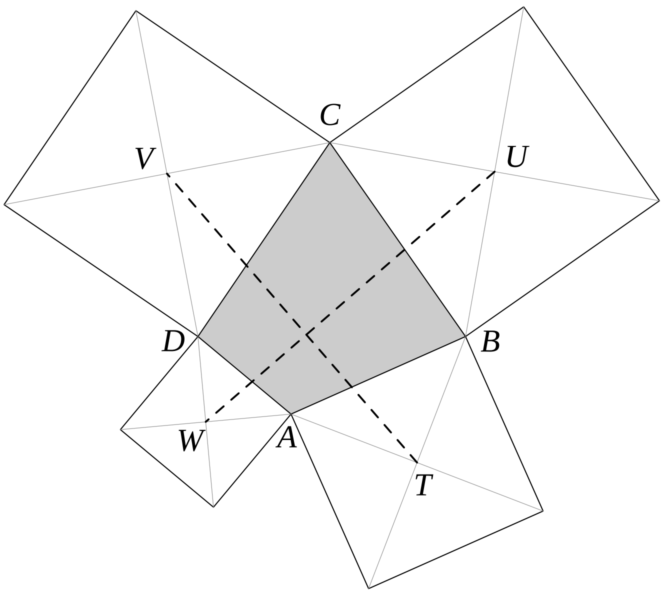
and

$$\begin{aligned} m - n &= \tfrac{1}{2}(c + d) - \tfrac{1}{2}(d + a) \\ &= \tfrac{1}{2}(c - a). \end{aligned}$$

Hence $l - k = m - n$, so that $\overrightarrow{KL} = \overrightarrow{NM}$ and therefore $KLMN$ is a parallelogram. ■

The appearance of squares in the statement of the following theorem, and the nature of the conclusion, mean that the result lends itself readily to a proof by complex numbers.

van Aubel's theorem *Given an arbitrary planar quadrilateral, place a square outwardly on each side. Then the two lines joining the centres of opposite squares are of equal length and perpendicular.*



Let the quadrilateral be $ABCD$ and the centres of the squares T, U, V, W, as shown.

Now A, T, B form three vertices of a square, so we may use the result of Example 5 above to obtain

$$t = \tfrac{1}{2}(a + b) + \tfrac{1}{2}i(a - b).$$

Similarly

$$\begin{aligned} u &= \tfrac{1}{2}(b + c) + \tfrac{1}{2}i(b - c), \\ v &= \tfrac{1}{2}(c + d) + \tfrac{1}{2}i(c - d), \\ w &= \tfrac{1}{2}(d + a) + \tfrac{1}{2}i(d - a). \end{aligned}$$

Hence

$$t - v = \tfrac{1}{2}(a + b - c - d) + \tfrac{1}{2}i(a - b - c + d)$$

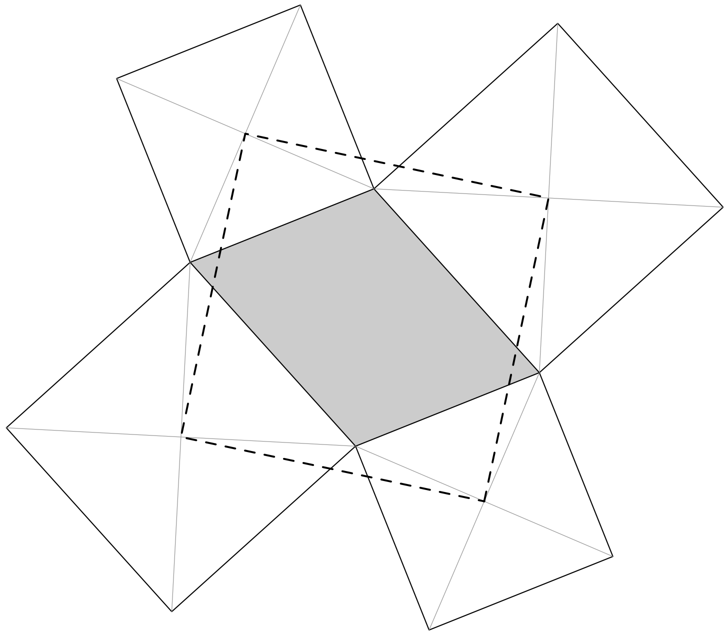
and

$$u - w = \tfrac{1}{2}(-a + b + c - d) + \tfrac{1}{2}i(a + b - c - d)$$
$$= i(t - v).$$

It follows that WU and VT are perpendicular and equal in length (see Example 3). ∎

The following result is really just a special case of van Aubel's theorem. We leave the reader to furnish a proof, and to answer the question: are van Aubel's theorem and Thébault's first theorem still true if the squares are placed *inwardly* on the sides, rather than outwardly?

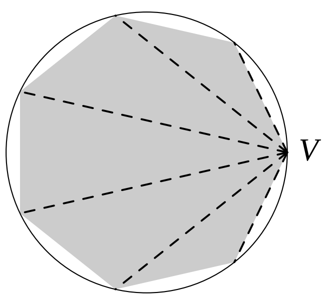
Thébault's first theorem *Given an arbitrary parallelogram, place a square outwardly on each side. Then the centres of these squares form a square.*



Regular polygons

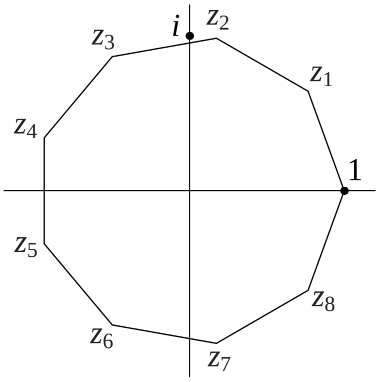
We conclude with a remarkable result about the diagonals of a regular polygon. The proof demonstrates the power of an approach by complex numbers, which brings into play a range of algebraic techniques.

Diagonals *For any integer $n \geq 3$, suppose that a regular polygon with n sides is inscribed in a circle of radius 1. Then the product of the lengths of the diagonals of the polygon passing through a given vertex V is equal to n. (Here the edges are counted as diagonals.)*



In order to keep the notation a little simpler, we present the proof for the case when $n = 9$, the nonagon. However, the proof is quite general.

Consider the nonagon in the complex plane with vertices at $1, z_1, z_2, \ldots, z_8$ on the unit circle centre 0, as shown in the figure below, and let V be the vertex at 1.



Now $1, z_1, z_2, \ldots, z_8$ have modulus 1 and arguments $2\pi s$, where $s = 0, \frac{1}{9}, \ldots, \frac{8}{9}$, and so may be written $e^{2\pi is}$ (that is, $\cos 2\pi s + i \sin 2\pi s$). But

$$(e^{2\pi is})^9 = e^{2\pi i(9s)} = 1$$

since $9s$ is an integer. Hence $1, z_1, z_2, \ldots, z_8$ are the roots of the equation $z^9 - 1 = 0$.

Since $z^9 - 1 \equiv (z-1)(z^8 + z^7 + \cdots + z + 1)$ it follows that $z_1, z_2, \ldots, z_8$ are the roots of the equation $z^8 + z^7 + \cdots + z + 1 = 0$. Hence

$$(z - z_1)(z - z_2)\cdots(z - z_8) \equiv z^8 + z^7 + \cdots + z + 1.$$

Putting $z = 1$ we obtain

$$(1 - z_1)(1 - z_2)\cdots(1 - z_8) = 9$$

and taking the modulus we get

$$|1 - z_1| \times |1 - z_2| \times \cdots \times |1 - z_8| = 9.$$

But $|1 - z_k|$ is equal to the distance from z_k to 1, that is, the length of one of the diagonals through V, so that the result follows. ∎