A graph with a unique Hamiltonian circuit

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The following problem was set by Robin Wilson at the 2009 National Mathematics Teachers' Summer School in York.

For the graph in Figure 1, can you visit each letter just once and return to your starting point? That is, does the graph have a Hamiltonian circuit?

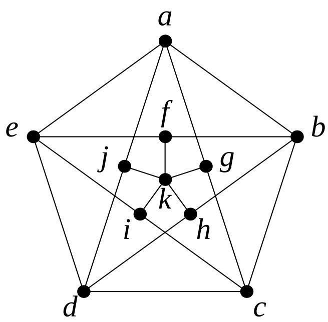


Figure 1

The answer is "yes"—the blue path in Figure 2 is one possible solution.

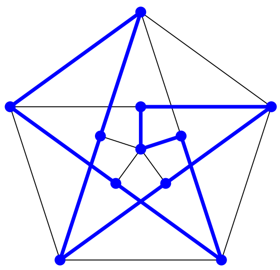
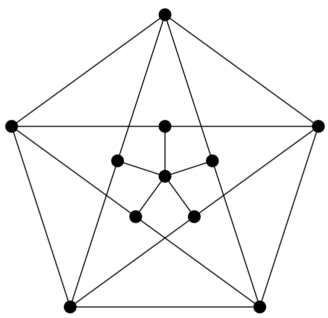


Figure 2

Clearly, other possible solutions may be obtained from this one by rotation or reflection. The question naturally arises, are these the only other solutions? That is, is the Hamiltonian circuit unique up to symmetry? In what follows we show that the answer to this question is also "yes".

Result *Up to symmetry, the graph below has a unique Hamiltonian circuit.*



Proof Consider the central vertex k. Up to symmetry, there are only two possible routes which a path can take through k, shown in blue in Figure 3. We deal with these two cases in turn.

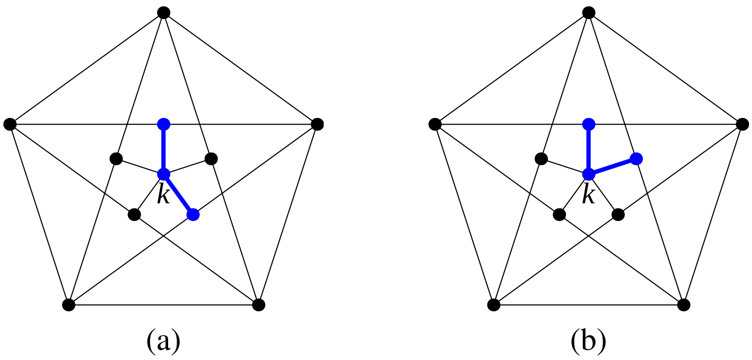
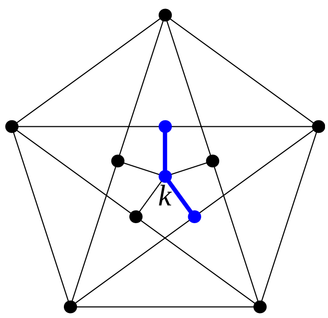


Figure 3

Case 1: Figure 3(a)



Since no other edges through k may be used we can omit these from the graph (Figure 4, left). We immediately see that there is only one route through each of vertices g, i and j (Figure 4, centre). The other edges at vertices a and c cannot now be used, so may also be omitted (Figure 4, right).

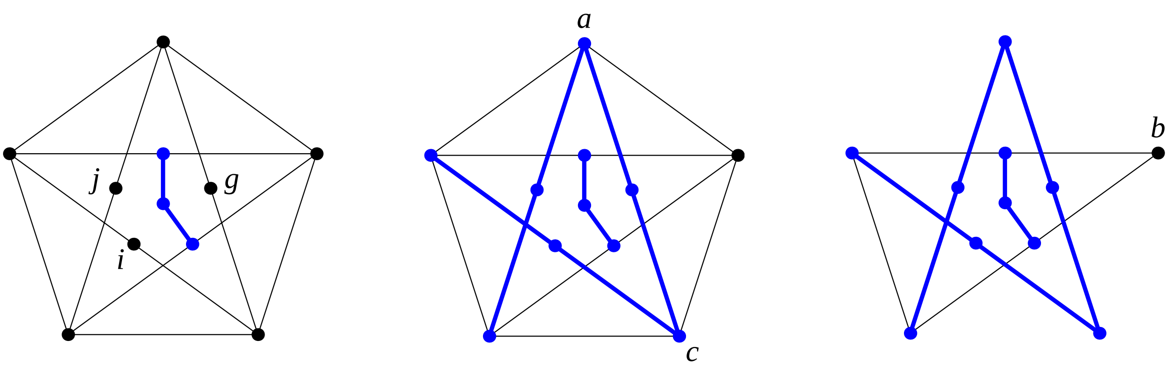
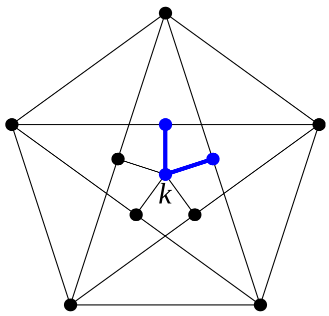


Figure 4

There is now a difficulty with vertex b, which cannot be joined to the rest of the path without forming a short circuit.

We conclude that no path is possible which connects the central vertex in the manner shown in Figure 3(a).

CASE 2: Figure 3(b)



Similarly to the first case, we may remove the other edges through k (Figure 5, left) and then deduce that there is only one route through vertices h, i and j (Figure 5, centre). We may also remove the other edges at vertex d (Figure 5, right).

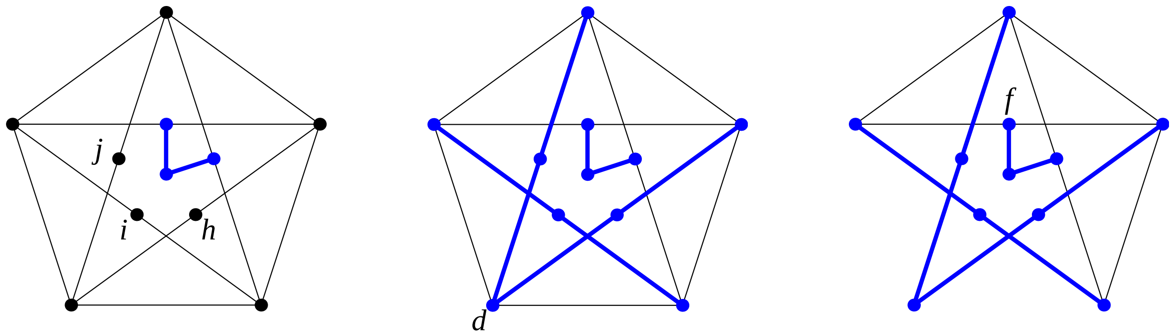


Figure 5

The path formed so far has three unconnected parts. However, all the vertices have been used, therefore the only way to complete a circuit is to use some of the remaining edges to connect these three parts like a chain. There are two ways to do this, determined by the choice of edge used at vertex f, as shown in Figure 6.

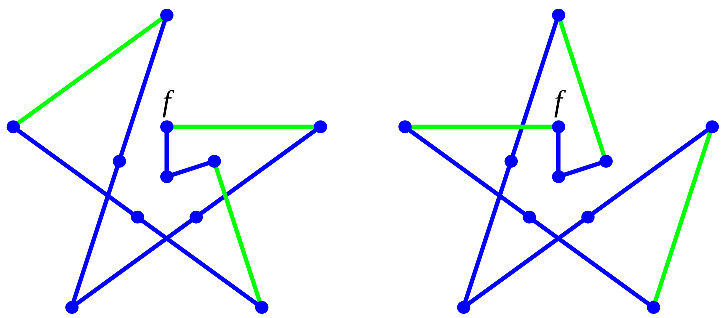


Figure 6

Note that the paths in Figure 6 are mirror images, so we conclude that there is only one way, up to symmetry, to form a circuit which connects the central vertex in the manner shown in Figure 3(b).

In summary, the Hamiltonian circuit shown in Figure 2 is unique up to symmetry. ∎