The central region of the Mandelbrot set

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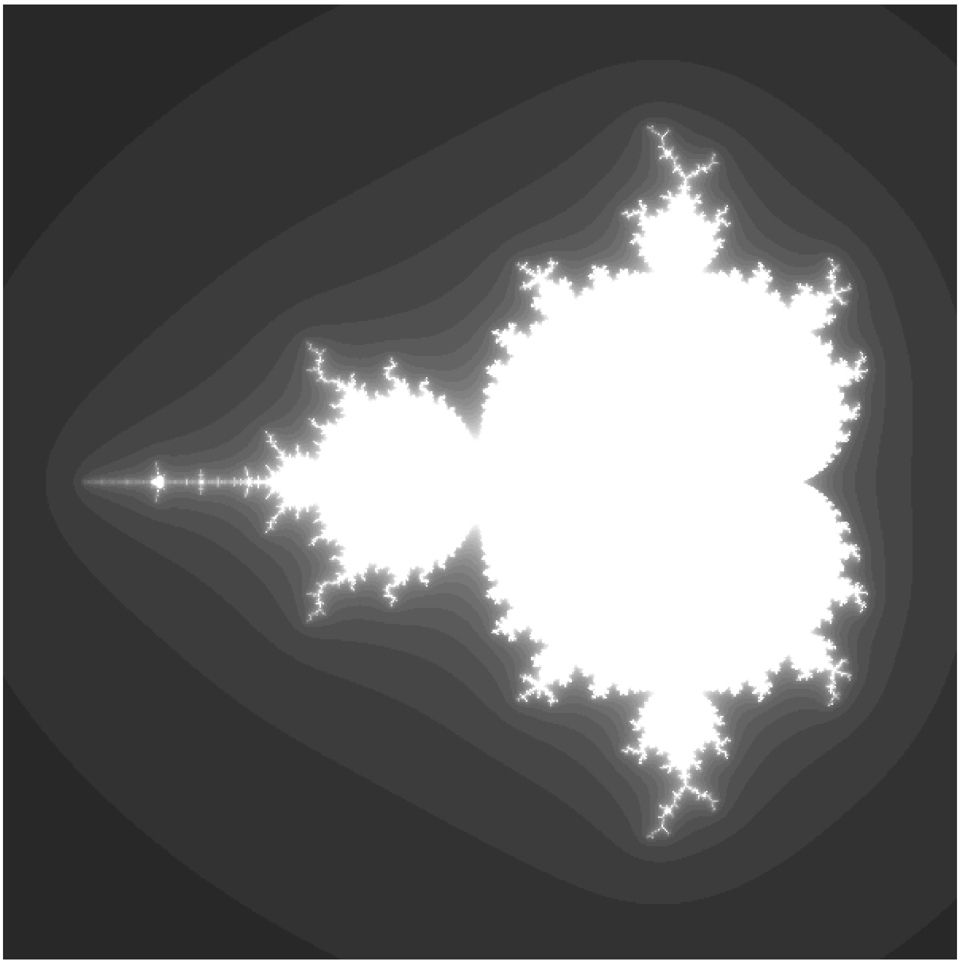

The Mandelbrot set shows values of the parameter c corresponding to mappings $f(z) = z^2 + c$ with certain properties. The largest, central region corresponds to mappings with an attracting fixed point. The region appears to have the shape of a cardioid, but is it actually one?

Result *The boundary of the central region of the Mandelbrot set is a cardioid.* □

If $f(z) = z^2 + c$ has a fixed point p, then $f(p) = p$, so that $p = p^2 + c$, or $c = p - p^2$. To be an attracting fixed point $|f'(p)| < 1$, so that $|2p| < 1$. Hence the central region contains values of c for which $c = p - p^2$ and $|p| < \frac{1}{2}$.

On the boundary of this region, $c = p - p^2$ and $|p| = \frac{1}{2}$. Referring to figure 1 on the following page, p lies on the blue circle, centre 0 radius $\frac{1}{2}$, and p^2 lies on the black circle, centre 0 radius $\frac{1}{4}$, with argument twice that of p. Hence $p - p^2$ lies on the circumference of the grey disc, centre p radius $\frac{1}{2}$, as shown, and the angle from $p - p^2$ to p to 0 is also equal to the argument of p.

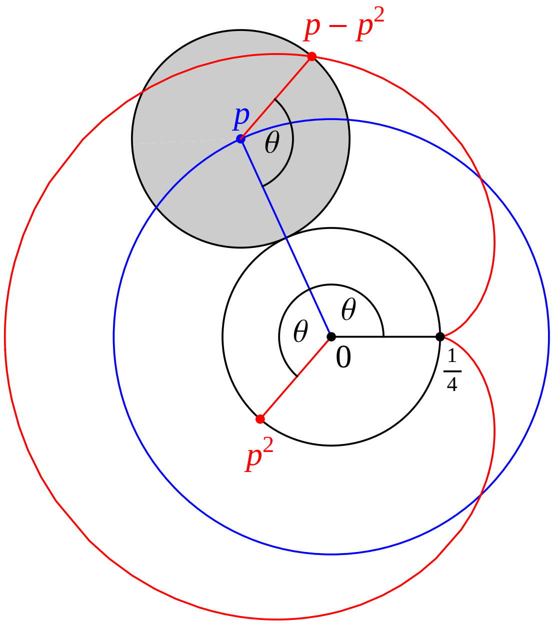

Figure 1

This corresponds precisely with the definition of a cardioid as an epicycloid—the locus of a point on the circumference of a disc as it rolls around an equal disc. Here the grey disc rolls around the black circle, starting with the point at $\frac{1}{4}$ and the centre of the rolling disc at $\frac{1}{2}$. ∎

An alternative derivation is possible by writing $p = \frac{1}{2}(\cos\theta + i\sin\theta)$ in order to find c and the equation of the curve.