

# Trigonometry and the nonagon

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The article *Three trigonometric results from a regular nonagon* by David Miles and Chris Pritchard in *Mathematics in School* (November 2008) stimulated me to consider whether the results discussed in the article were true more generally. It turns out that each of them is indeed related to a more general result, though in very different ways.

For reasons that will become clear we consider the results in a different order to the original article.

## Morrie's Law

### Result 1

For any angle  $A$ ,

$$8 \cos A \cos 2A \cos 4A = \frac{\sin 8A}{\sin A}. \quad \square$$

This generalises the standard formula  $\sin 2A = \sin A \cos A$  and is itself a special case of a more general result.

To prove Result 1, note that

$$\begin{aligned} \sin 8A &= 2 \sin 4A \cos 4A \\ &= 2 \times 2 \sin 2A \cos 2A \cos 4A \\ &= 2 \times 2 \times 2 \sin A \cos A \cos 2A \cos 4A. \quad \blacksquare \end{aligned}$$

Putting  $A = 20^\circ$  in Result 1 we obtain

$$\begin{aligned} 8 \cos 20^\circ \cos 40^\circ \cos 80^\circ &= \frac{\sin 160^\circ}{\sin 20^\circ} \\ &= \frac{\sin 20^\circ}{\sin 20^\circ} \\ &= 1, \quad (1) \end{aligned}$$

from which Morrie's Law follows.

## A Tangent Product

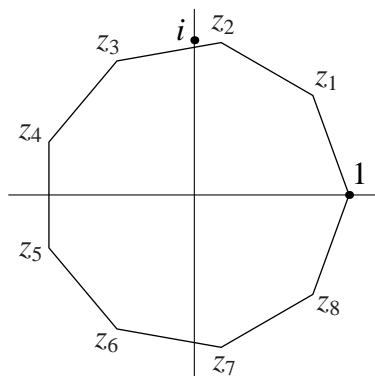
Result 1 concerns a product of cosines. We now derive a similar result for a product of sines by considering the diagonals of a regular polygon. The corresponding result for tangents then follows by division.

### Result 2

For any integer  $n \geq 3$ , suppose that a regular polygon with  $n$  sides is inscribed in a circle of radius 1. Then the product of the lengths of the diagonals of the polygon passing through a given vertex  $V$  is equal to  $n$ . (For our purposes here, the edges are also counted as diagonals.)  $\square$

We shall use Result 2 in the case when  $n = 9$ , the nonagon, so we present the proof for this case. However, the proof is quite general.

Consider the nonagon with vertices at  $1, z_1, z_2, \dots, z_8$  in the complex plane, as shown in the figure, and let  $V$  be the vertex at  $1$ .



Now  $1, z_1, z_2, \dots, z_8$  may be written  $e^{2\pi is}$ , where  $s = 0, \frac{1}{9}, \dots, \frac{8}{9}$ . Hence  $1, z_1, z_2, \dots, z_8$  are the roots of the equation  $z^9 - 1 = 0$ .

Since  $z^9 - 1 \equiv (z - 1)(z^8 + z^7 + \dots + z + 1)$  it follows that  $z_1, z_2, \dots, z_8$  are the roots of the equation  $z^8 + z^7 + \dots + z + 1 = 0$ . Hence

$$(z - z_1)(z - z_2) \cdots (z - z_8) \equiv z^8 + z^7 + \dots + z + 1.$$

Putting  $z = 1$  we obtain

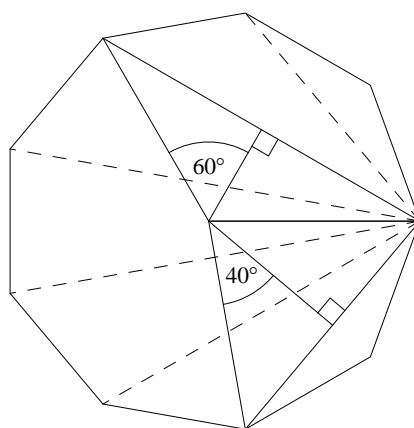
$$(1 - z_1)(1 - z_2) \cdots (1 - z_8) = 9$$

and taking the modulus we get

$$|1 - z_1| |1 - z_2| \cdots |1 - z_8| = 9.$$

But  $|1 - z_k|$  is equal to the distance from  $1$  to  $z_k$ , that is, the length of a diagonal from  $V$ , so that Result 2 follows. ■

For a nonagon the diagonals occur in pairs and have lengths  $2 \sin 20^\circ, 2 \sin 40^\circ, 2 \sin 60^\circ$  and  $2 \sin 80^\circ$ , as indicated in the figure.



Using Result 2 we obtain

$$(2 \sin 20^\circ)(2 \sin 40^\circ)(2 \sin 60^\circ)(2 \sin 80^\circ) = \sqrt{9}.$$

Since  $\sin 60^\circ = \frac{\sqrt{3}}{2}$  this simplifies to give

$$8 \sin 20^\circ \sin 40^\circ \sin 80^\circ = \sqrt{3}. \quad (2)$$

Dividing equation (2) by equation (1) we obtain

$$\begin{aligned} \tan 20^\circ \tan 40^\circ \tan 80^\circ &= \sqrt{3} \\ &= \tan 60^\circ, \end{aligned} \quad (3)$$

as required.

## The Tangent Sum Challenge

The results about a product and a sum of tangents, given separately in the original article, are in fact directly related. To see this we make use of the following general result connecting a sum and a product of tangents.

### Result 3

*If  $A$ ,  $B$  and  $C$  are the angles in a triangle—that is, if  $A + B + C = 180^\circ$ —then*

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C. \quad \square$$

Result 3 follows immediately from the tangent formula

$$\tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

and the fact that  $\tan 180^\circ = 0$ . (The tangent formula for three angles is a generalisation of the better known formula for  $\tan(A + B)$  and may be derived from it.) ■

Putting  $A = 70^\circ$ ,  $B = 60^\circ$  and  $C = 50^\circ$  in Result 3 we obtain

$$\tan 70^\circ + \tan 60^\circ + \tan 50^\circ = \tan 70^\circ \tan 60^\circ \tan 50^\circ. \quad (4)$$

Now from equation (3)

$$\tan 20^\circ \tan 40^\circ \tan 80^\circ = \tan 60^\circ$$

so that

$$\cot 70^\circ \cot 50^\circ \tan 80^\circ = \tan 60^\circ,$$

which rearranges to give

$$\begin{aligned} \tan 80^\circ &= \tan 70^\circ \tan 60^\circ \tan 50^\circ \\ &= \tan 70^\circ + \tan 60^\circ + \tan 50^\circ, \end{aligned}$$

from equation (4), as required.