Trigonometry and the nonagon

Andrew Jobbings
www.arbelos.co.uk

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The article *Three trigonometric results from a regular nonagon* by David Miles and Chris Pritchard in *Mathematics in School* (November 2008) stimulated me to consider whether the results discussed in the article were true more generally. It turns out that each of them is indeed related to a more general result, though in very different ways.

For reasons that will become clear we consider the results in a different order to the original article.

Morrie's Law

Result 1
For any angle A,

$$8\cos A\cos 2A\cos 4A = \frac{\sin 8A}{\sin A}. \qquad \square$$

This generalises the standard formula $\sin 2A = \sin A\cos A$ and is itself a special case of a more general result.

To prove Result 1, note that

$$\begin{aligned}\sin 8A &= 2\sin 4A\cos 4A\\ &= 2\times 2\sin 2A\cos 2A\cos 4A\\ &= 2\times 2\times 2\sin A\cos A\cos 2A\cos 4A. \qquad \blacksquare\end{aligned}$$

Putting $A = 20^\circ$ in Result 1 we obtain

$$\begin{aligned}8\cos 20^\circ\cos 40^\circ\cos 80^\circ &= \frac{\sin 160^\circ}{\sin 20^\circ}\\ &= \frac{\sin 20^\circ}{\sin 20^\circ}\\ &= 1, \qquad (1)\end{aligned}$$

from which Morrie's Law follows.

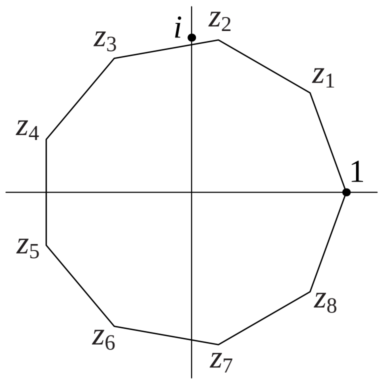
A Tangent Product

Result 1 concerns a product of cosines. We now derive a similar result for a product of sines by considering the diagonals of a regular polygon. The corresponding result for tangents then follows by division.

Result 2
For any integer $n \geq 3$, suppose that a regular polygon with n sides is inscribed in a circle of radius 1. Then the product of the lengths of the diagonals of the polygon passing through a given vertex V is equal to n. (For our purposes here, the edges are also counted as diagonals.) $\square$

We shall use Result 2 in the case when $n = 9$, the nonagon, so we present the proof for this case. However, the proof is quite general.

Consider the nonagon with vertices at $1, z_1, z_2, \ldots, z_8$ in the complex plane, as shown in the figure, and let V be the vertex at 1.



Now $1, z_1, z_2, \ldots, z_8$ may be written $e^{2\pi i s}$, where $s = 0, \frac{1}{9}, \ldots, \frac{8}{9}$. Hence $1, z_1, z_2, \ldots, z_8$ are the roots of the equation $z^9 - 1 = 0$.

Since $z^9 - 1 \equiv (z-1)(z^8 + z^7 + \cdots + z + 1)$ it follows that $z_1, z_2, \ldots, z_8$ are the roots of the equation $z^8 + z^7 + \cdots + z + 1 = 0$. Hence

$$(z - z_1)(z - z_2)\cdots(z - z_8) \equiv z^8 + z^7 + \cdots + z + 1.$$

Putting $z = 1$ we obtain

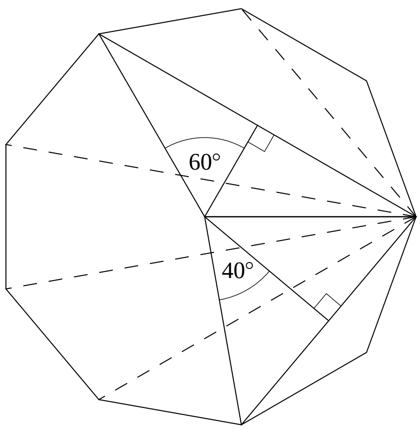
$$(1 - z_1)(1 - z_2)\cdots(1 - z_8) = 9$$

and taking the modulus we get

$$|1 - z_1|\,|1 - z_2|\cdots|1 - z_8| = 9.$$

But $|1 - z_k|$ is equal to the distance from 1 to z_k, that is, the length of a diagonal from V, so that Result 2 follows. ■

For a nonagon the diagonals occur in pairs and have lengths $2\sin 20°$, $2\sin 40°$, $2\sin 60°$ and $2\sin 80°$, as indicated in the figure.



Using Result 2 we obtain

$$(2\sin 20^\circ)(2\sin 40^\circ)(2\sin 60^\circ)(2\sin 80^\circ) = \sqrt{9}.$$

Since $\sin 60^\circ = \frac{\sqrt{3}}{2}$ this simplifies to give

$$8\sin 20^\circ \sin 40^\circ \sin 80^\circ = \sqrt{3}. \tag{2}$$

Dividing equation (2) by equation (1) we obtain

$$\begin{aligned}\tan 20^\circ \tan 40^\circ \tan 80^\circ &= \sqrt{3} \\ &= \tan 60^\circ,\end{aligned} \tag{3}$$

as required.

The Tangent Sum Challenge

The results about a product and a sum of tangents, given separately in the original article, are in fact directly related. To see this we make use of the following general result connecting a sum and a product of tangents.

Result 3
If A, B and C are the angles in a triangle—that is, if $A + B + C = 180^\circ$—then

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C.$$ □

Result 3 follows immediately from the tangent formula

$$\tan(A + B + C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

and the fact that $\tan 180^\circ = 0$. (The tangent formula for three angles is a generalisation of the better known formula for $\tan(A + B)$ and may be derived from it.) ■

Putting $A = 70^\circ$, $B = 60^\circ$ and $C = 50^\circ$ in Result 3 we obtain

$$\tan 70^\circ + \tan 60^\circ + \tan 50^\circ = \tan 70^\circ \tan 60^\circ \tan 50^\circ. \tag{4}$$

Now from equation (3)

$$\tan 20^\circ \tan 40^\circ \tan 80^\circ = \tan 60^\circ$$

so that

$$\cot 70^\circ \cot 50^\circ \tan 80^\circ = \tan 60^\circ,$$

which rearranges to give

$$\begin{aligned}\tan 80^\circ &= \tan 70^\circ \tan 60^\circ \tan 50^\circ \\ &= \tan 70^\circ + \tan 60^\circ + \tan 50^\circ,\end{aligned}$$

from equation (4), as required.