Parity

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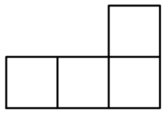
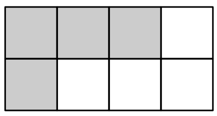
A tiling problem

European Kangaroo 1999, Question 16
Which size of rectangle cannot be made using tiles of the form shown?

A 4×4 B 6×6 C 8×8 D 4×6 E 6×8



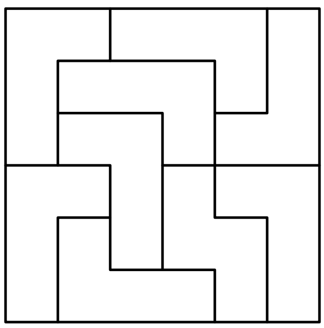
The printed UKMT solution for the above problem clearly explains how to tile the rectangles given in options A, C, D and E. Two tiles fit together to make a 2×4 large tile and so we can make each of the rectangles in A, C, D and E by fitting together these 2×4 large tiles.



The printed solution is less convincing in showing that the 6×6 rectangle cannot be tiled. Of course, when attempting a multiple-choice contest like the Kangaroo this is not necessary, since the other four options may be eliminated. However, no mathematician would be satisfied until it had been proved that tiling the 6×6 rectangle was not possible.

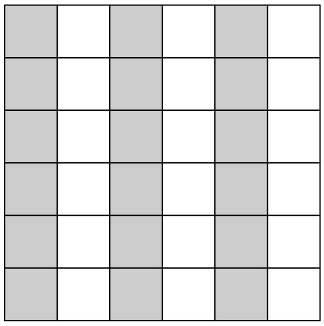
It is not immediately clear how to go about proving something to be impossible—most proofs show that something *is* the case. The proof given below involves the idea of parity, which is a useful one in various problems.

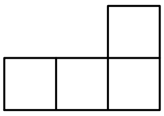
In order to understand the nature of the problem, consider the following diagram, which shows an attempt at a tiling. Can you see where it has failed? That is easy, but perhaps you can modify the diagram so that it works? After failing to do that, the difficult part is then to explain why it is not possible—repeated failure is not a proof!



Using a parity argument

To prove that the tiling is impossible we colour the small squares of the 6×6 rectangle grey or white, as shown in the following diagram.





There are 36 squares altogether, and each tile covers 4 squares, so that 9 tiles are needed. Now consider what happens when one of the tiles is placed on the rectangle. Whichever way that a tile is placed, you will notice that the tile covers 3 squares of one colour and 1 square of the other colour. Each of these is an odd number, so using 9 tiles will cover an odd number of squares of each colour. But there are 18 squares of each colour on the board. Hence it is impossible to tile the rectangle.

Another problem

Here is another problem where a parity proof may be used, which you may like to try.

The two opposite squares of a chessboard are removed, as shown.

Is it possible to cover the remaining shape with dominoes, each of which covers two squares of the chessboard?