Playing with Dice

ANDREW JOBBINGS
www.arbelos.co.uk

18 December 2000

Two normal dice

With two normal dice, each labelled from 1 to 6, it is easily established that the probabilities of achieving the possible scores (obtained by adding the numbers on the dice) are:

Score	2 or 12	3 or 11	4 or 10	5 or 9	6 or 8	7
Probability	$\frac{1}{36}$	$\frac{1}{18}$	$\frac{1}{12}$	$\frac{1}{9}$	$\frac{5}{36}$	$\frac{1}{6}$

One way to do this is to draw a possibility space diagram. Alternatively, just list the different ways in which each score may be obtained.

So, for example, a score of 5 may be obtained as $1+4$, $2+3$, $3+2$, or $4+1$. This means that 5 may be scored in four out of a total of thirty-six possible ways, giving a probability of $\frac{1}{9}$.

Note that the order of the numbers on the two dice is important: the two dice may be considered to have different colours, say blue and red, with the blue number listed first.

Unusual dice

Two dice

Now consider two abnormal dice, labelled as follows:

Blue	1 3 4 5 6 8
Red	1 2 2 3 3 4

What are the possible scores and probabilities for these dice?

It is readily seen that all the usual scores from 2 to 12 may be obtained. More surprisingly, it turns out that these scores also have the same probabilities as for two normal dice.

For example, a score of 5 may be obtained as $1+4$, $3+2$, $3+2$, or $4+1$, where the blue die is listed first. The outcome $3+2$ is repeated since the red die is labelled twice with the number 2. Once again 5 may be scored in four out of a total of thirty-six possible ways, giving the same probability as before. All the other scores may be checked in a similar way, or using a possibility space diagram.

A coin and a die

More surprising still, consider a coin and an eighteen-sided die with the following numbers:

Coin	1 2
Die	1 2 3 3 4 4 5 5 5 6 6 6 7 7 8 8 9 10

Can you guess what the scores and probabilities are in this case? You are invited to check for yourself.

Three non-transitive dice

Finally, consider the following three dice:

Amber	3 4 8
Green	1 5 9
Red	2 6 7

Suppose that Amber is rolled against Green and the higher number wins. Of the nine possible outcomes, four are favourable to Amber (3–1, 4–1, 8–1, or 8–5) and five to Green. So, on average, Green wins more than Amber does and we may say that Green beats Amber.

Similarly, if Green is rolled against Red, we find that Red beats Green by five outcomes to four.

Red beats Green and Green beats Amber. Does this imply that Red beats Amber? Not at all! These dice are *non-transitive*. If you check, you will find that Amber beats Red by five outcomes to four.

Though such non-transitivity is familiar in areas of human endeavour, like sports, it is more surprising in an area of mathematics where we have complete information. Similar effects come into play in election schemes where there are more than two candidates.

References

Brian C Fowler and Randall J Swift. *Relabeling dice*. The College Mathematics Journal, 30(3):204–208, May 1999.

Donald G Saari and Fabrice Valognes. *Geometry, Voting, and Paradoxes*. Mathematics Magazine, 71(4):243–259, October 1998.