A generalisation of Pythagoras’ theorem

ANDREW JOBBINGS

www.arbelos.co.uk

23 December 2008

This result was discussed in SYMmetryplus, Issue No 10, Autumn 1999, The proof given there made use of Hero’s formula for the area of a triangle. The proof given below only makes use of the standard version of Pythagoras’ theorem.

Theorem Given a tetrahedron in which the three angles at one vertex are all right angles, as shown above, let $A_1$, $A_2$ and $A_3$ be the areas of the three right-angled faces and let $A$ be the area of the fourth triangular face. Then

$$A^2 = A_1^2 + A_2^2 + A_3^2.$$  

Label the tetrahedron $OPQR$, where $O$ is the vertex which has three right angles, as shown below, and let $F$ be the foot of the perpendicular from $P$ to $QR$.

Since the edge $OP$ is perpendicular to the edges $OQ$ and $OR$, it is perpendicular to the face $OQR$, and hence is perpendicular to $OF$. In other words, the triangle $OFT$ is right-angled.

Also, $OP$ is perpendicular to $QR$. Hence $QR$ is perpendicular to both $PF$ and $OP$, so it is perpendicular to $OF$. Therefore $OF$ is a height of triangle $OQR$. 
Let the lengths of four sides of the tetrahedron be \(a, b, c\) and \(\alpha\), as shown above, and let the perpendicularly have lengths \(h\) and \(H\).

\[
\alpha
\]
\[
h
\]
\[
H
\]

Now

\[
A^2 = \left(\frac{1}{2} \alpha H\right)^2
\]
\[
= \frac{1}{4} \alpha^2 H^2.
\]

Apply Pythagoras' theorem to the triangle with sides \(a, h\) and \(H\) to obtain

\[
A^2 = \frac{1}{4} \alpha^2 \left(h^2 + a^2\right)
\]
\[
= \frac{1}{4} \alpha^2 h^2 + \frac{1}{4} \alpha^2 a^2.
\]

Now apply Pythagoras' theorem to the triangle with sides \(b, c\) and \(\alpha\) to obtain

\[
A^2 = \frac{1}{4} \alpha^2 h^2 + \frac{1}{4} \left(b^2 + c^2\right) a^2
\]
\[
= \frac{1}{4} \alpha^2 h^2 + \frac{1}{4} \alpha^2 b^2 + \frac{1}{4} \alpha^2 c^2
\]
\[
= \left(\frac{1}{2} \alpha h\right)^2 + \left(\frac{1}{2} \alpha b\right)^2 + \left(\frac{1}{2} \alpha c\right)^2
\]
\[
= A_1^2 + A_2^2 + A_3^2.
\]