A generalisation of Pythagoras' theorem

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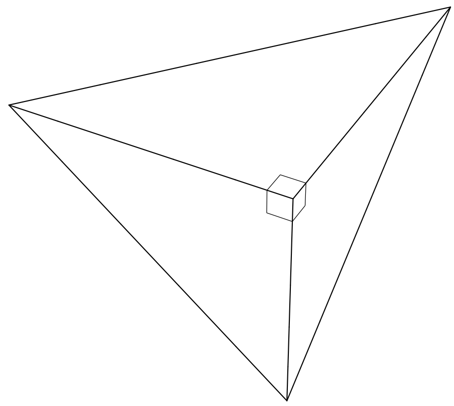
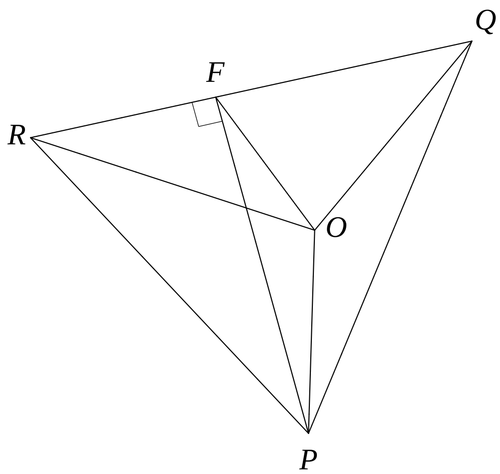
This result was discussed in SYMmetry*plus*, Issue No 10, Autumn 1999, The proof given there made use of Hero's formula for the area of a triangle. The proof given below only makes use of the standard version of Pythagoras' theorem.



Theorem *Given a tetrahedron in which the three angles at one vertex are all right angles, as shown above, let* A_1, A_2 *and* A_3 *be the areas of the three right-angled faces and let* A *be the area of the fourth triangular face. Then*

$$A^2 = A_1^2 + A_2^2 + A_3^2. \qquad \square$$

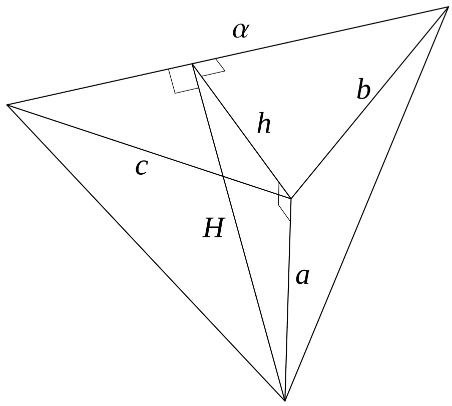
Label the tetrahedron $OPQR$, where O is the vertex which has three right angles, as shown below, and let F be the foot of the perpendicular from P to QR.



Since the edge OP is perpendicular to the edges OQ and OR, it is perpendicular to the face OQR, and hence is perpendicular to OF. In other words, the triangle OFT is right-angled.

Also, OP is perpendicular to QR. Hence QR is perpendicular to both PF and OP, so it is perpendicular to OF. Therefore OF is a height of triangle OQR.

Let the lengths of four sides of the tetrahedron be a, b, c and α, as shown above, and let the perpendiculars have lengths h and H.



Now

$$\begin{aligned} A^2 &= \left(\tfrac{1}{2}\alpha H\right)^2 \\ &= \tfrac{1}{4}\alpha^2 H^2. \end{aligned}$$

Apply Pythagoras' theorem to the triangle with sides a, h and H to obtain

$$\begin{aligned} A^2 &= \tfrac{1}{4}\alpha^2\left(h^2 + a^2\right) \\ &= \tfrac{1}{4}\alpha^2 h^2 + \tfrac{1}{4}\alpha^2 a^2. \end{aligned}$$

Now apply Pythagoras' theorem to the triangle with sides b, c and α to obtain

$$\begin{aligned} A^2 &= \tfrac{1}{4}\alpha^2 h^2 + \tfrac{1}{4}\left(b^2 + c^2\right) a^2 \\ &= \tfrac{1}{4}\alpha^2 h^2 + \tfrac{1}{4}a^2 b^2 + \tfrac{1}{4}a^2 c^2 \\ &= \left(\tfrac{1}{2}\alpha h\right)^2 + \left(\tfrac{1}{2}ab\right)^2 + \left(\tfrac{1}{2}ac\right)^2 \\ &= A_1^2 + A_2^2 + A_3^2. \end{aligned}$$

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