

Deriving the quadratic formula

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The ‘standard’ derivation of the quadratic formula involves completing the square. One way to avoid this is to make use of the connection between the roots and coefficients.

Vieta’s formulae *Suppose the roots of $ax^2 + bx + c = 0$ are α, β . Then*

$$\alpha + \beta = -\frac{b}{a} \quad \text{and} \quad \alpha\beta = \frac{c}{a}.$$

Since α and β are the roots of $ax^2 + bx + c = 0$, $x - \alpha$ and $x - \beta$ are factors of $ax^2 + bx + c$. Hence

$$\begin{aligned} ax^2 + bx + c &\equiv a(x - \alpha)(x - \beta) \\ &\equiv ax^2 - a(\alpha + \beta)x + a\alpha\beta. \end{aligned}$$

By comparing the coefficients, we deduce that $b = a(\alpha + \beta)$ and $c = a\alpha\beta$, from which the required results follow. ■

The quadratic formula *The solutions of $ax^2 + bx + c = 0$ are*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We first find an expression for $\alpha - \beta$ using both Vieta’s formulae and then use this with Vieta’s expression for $\alpha + \beta$ to find α .

Now

$$\begin{aligned} (\alpha - \beta)^2 &\equiv (\alpha + \beta)^2 - 4\alpha\beta \\ &= \frac{b^2}{a^2} - 4\frac{c}{a} \\ &= \frac{b^2 - 4ac}{a^2}. \end{aligned}$$

Therefore

$$\alpha - \beta = \frac{\pm \sqrt{b^2 - 4ac}}{a}.$$

Hence

$$\begin{aligned} \alpha &\equiv \frac{1}{2} [(\alpha + \beta) + (\alpha - \beta)] \\ &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \end{aligned} \quad \blacksquare$$