Deriving the quadratic formula

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12 May 2006

The 'standard' derivation of the quadratic formula involves completing the square. One way to avoid this is to make use of the connection between the roots and coefficients.

Vieta's formulae *Suppose the roots of* $ax^2 + bx + c = 0$ *are* α, β. *Then*

$$\alpha + \beta = -\frac{b}{a} \quad \textit{and} \quad \alpha\beta = \frac{c}{a}.$$

Since α and β are the roots of $ax^2 + bx + c = 0$, $x - \alpha$ and $x - \beta$ are factors of $ax^2 + bx + c$. Hence

$$\begin{aligned} ax^2 + bx + c &\equiv a(x - \alpha)(x - \beta) \\ &\equiv ax^2 - a(\alpha + \beta)x + a\alpha\beta. \end{aligned}$$

By comparing the coefficients, we deduce that $b = a(\alpha + \beta)$ and $c = a\alpha\beta$, from which the required results follow. ∎

The quadratic formula *The solutions of* $ax^2 + bx + c = 0$ *are*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

We first find an expression for $\alpha - \beta$ using both Vieta's formulae and then use this with Vieta's expression for $\alpha + \beta$ to find α.

Now

$$\begin{aligned} (\alpha - \beta)^2 &\equiv (\alpha + \beta)^2 - 4\alpha\beta \\ &= \frac{b^2}{a^2} - 4\frac{c}{a} \\ &= \frac{b^2 - 4ac}{a^2}. \end{aligned}$$

Therefore

$$\alpha - \beta = \frac{\pm\sqrt{b^2 - 4ac}}{a}.$$

Hence

$$\begin{aligned} \alpha &\equiv \tfrac{1}{2}\left[(\alpha + \beta) + (\alpha - \beta)\right] \\ &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \end{aligned}$$

∎