Rings of regular polygons

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*In memory of Leon van den Broek—an inspiration to us all.**

Suppose a number of congruent regular polygons are placed in a ring, such as those shown in figure 1.

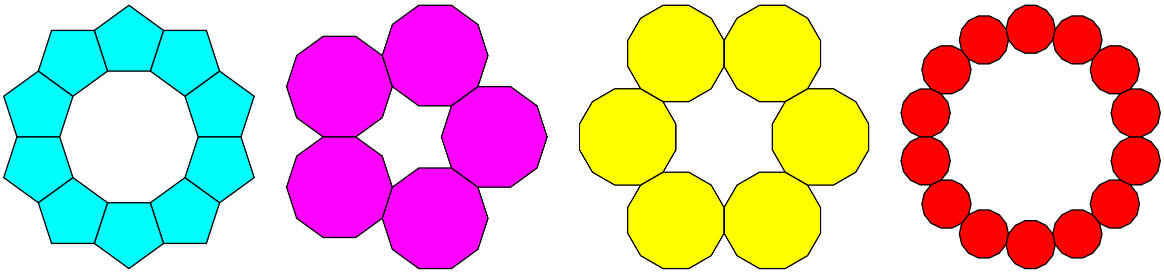


Figure 1

When the boundary of the region inside the ring consists of exactly k edges of each polygon, we refer to the configuration as a k-*ring*. Figure 1 shows, from left to right, ten pentagons forming a 1-ring, five decagons forming a 2-ring, six dodecagons forming a 3-ring, and fourteen 14-gons forming a 5-ring. Provided we allow the region inside the ring to be empty, it is quite possible to have a 0-ring; in that case the polygons are just arranged around a point.

The question naturally arises, how many different k-rings are there for any given k?

Result
The number of k-rings is equal to the number of factors of $4(k+1)$.

PROOF Suppose we have a ring of m regular n-gons. Note that two k-rings with different ordered pairs (m, n) are different; otherwise they are considered to be the same (in other words, we ignore things like scale).

Let the centre of the ring be O.

Consider one polygon in the ring, with centre C and sharing edges AB and $A'B'$ with the two immediate neighbours in the ring (see figure 2). There are $k+1$ vertices of the polygon between A' and A (inclusive), indicated by the dotted points.

We may divide the polygon into n equal parts by joining the midpoint of each edge to C; let the angle at C in each part be $\alpha°$, and let $\angle AOA' = \beta°$, as shown in figure 2.

From the angle sum of quadrilateral $OMCM'$ we have

$$\beta + 90 + (k+1)\alpha + 90 = 360.$$

But $n\alpha = 360$ and $m\beta = 360$, so that

$$\frac{360}{m} + \frac{360(k+1)}{n} = 180.$$

*This paper was prompted by an image that Leon sent to me in December 2012. The image is reproduced on page 3. Leon died unexpectedly on 11 December 2013.

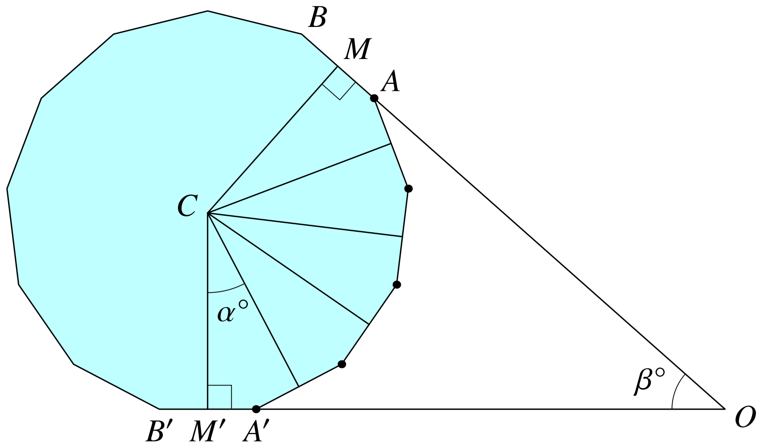


Figure 2

Dividing each term by 180, we get

$$\frac{2}{m} + \frac{2(k+1)}{n} = 1.$$

Now multiplying each term by mn we obtain

$$2n + 2(k+1)m = mn,$$

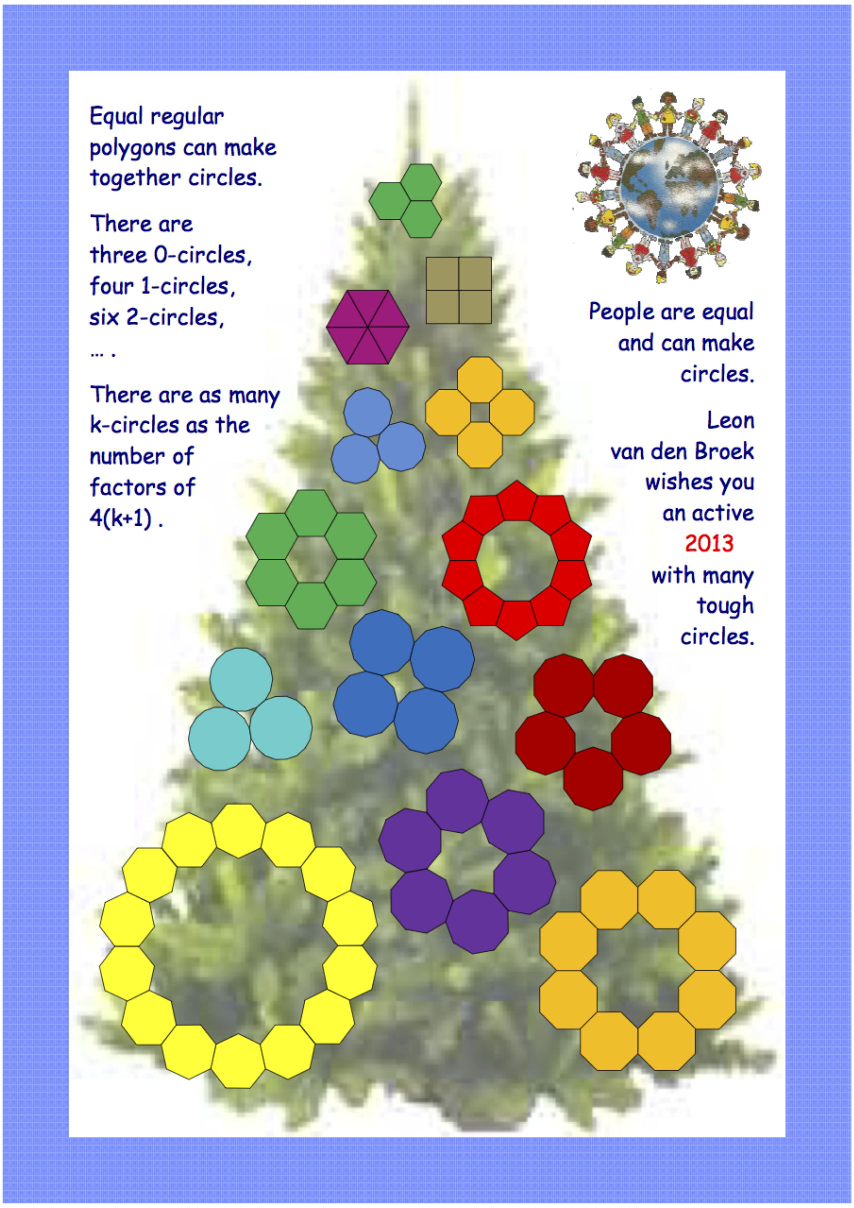
which we may rewrite in the form

$$4(k+1) = (m-2)\big(n - 2(k+1)\big).$$

Therefore each ordered factorisation of $4(k+1)$ corresponds to a solution for the integers m and n. Hence the number of ordered pairs (m, n) is equal to the number of factors of $4(k+1)$.■

The factorisations of $4(k+1)$ corresponding to the examples in figure 1 are:

$$\begin{aligned} 8 \times 1 &= (10-2)(5-4); \\ 3 \times 4 &= (5-2)(10-6); \\ 4 \times 4 &= (6-2)(12-8); \\ \text{and} \quad 12 \times 2 &= (14-2)(14-12). \end{aligned}$$


Equal regular polygons can make together circles.
There are three 0-circles, four 1-circles, six 2-circles, … .
There are as many k-circles as the number of factors of 4(k+1) .
People are equal and can make circles.
Leon van den Broek wishes you an active 2013 with many tough circles.