Langley's 20-80-80 isosceles triangle problem

ANDREW JOBBINGS
www.arbelos.co.uk

15 December 2004

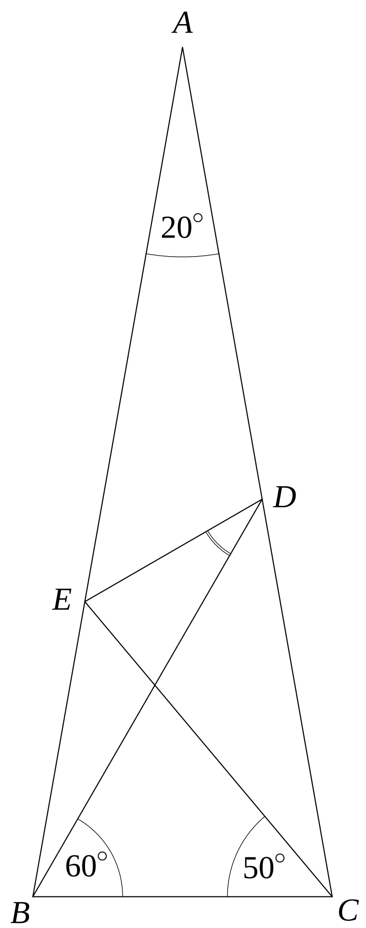
The problem is a classic, named after Edward M Langley, who set it in *The Mathematical Gazette*, Vol. 11, No. 160 (Oct., 1922), p. 173. However, the problem is older than this; it appears, for example, in a Cambridge college scholarship examination of 1916.

There are many solutions—we give here what is perhaps the neatest solution, which involves making a far from obvious construction.

Langley's problem

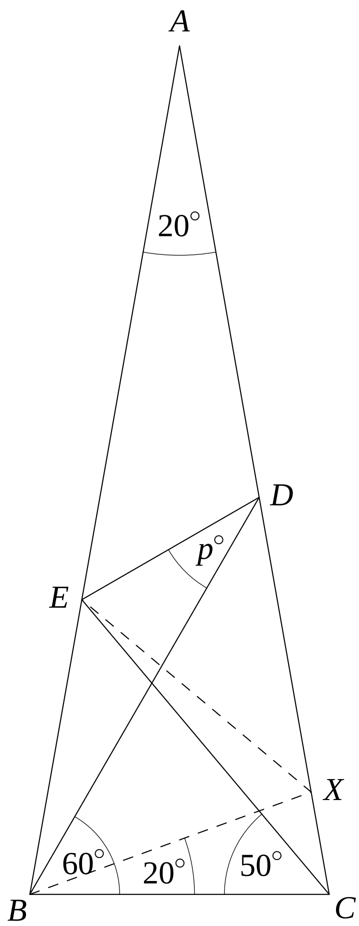
In triangle ABC, $\angle A = 20^\circ$ and $AB = AC$. The point D lies on AC so that $\angle DBC = 60^\circ$ and the point E lies on AB so that $\angle ECB = 50^\circ$.

Find the size of $\angle BDE$.



Solution

Construct X on CD so that $\angle XBC = 20^\circ$, as shown in the following figure.



Then

$$\begin{aligned} EB &= BC \quad (\triangle EBC \text{ has equal base angles}) \\ &= BX \quad (\triangle BCX \text{ has equal base angles}). \end{aligned}$$

Hence $\triangle EBX$ is equilateral ($EB = BX$ and $\angle EBX = 60^\circ$) and

$$\begin{aligned} EX &= BX \quad (\text{equilateral triangle}) \\ &= XD \quad (\triangle BXD \text{ has equal base angles}). \end{aligned}$$

Therefore $\triangle EXD$ is isosceles.

But $\angle EXD = 180^\circ - \angle EXB - \angle BXC = 180^\circ - 60^\circ - 80^\circ = 40^\circ$, so that $\angle XDE = 70^\circ$.

Since $p^\circ = \angle XDE - \angle XDB$, we conclude that $p = 70 - 40 = 30$.